DOI: https://doi.org/10.15688/mpcm.jvolsu.2022.1.2

UDC 514.7
LBC 22.151

Submitted: 14.02.2021
Accepted: 02.02.2022

k-YAMABE AND QUASI k-YAMABE SOLITONS ON IMPERFECT FLUID GENERALIZED ROBERTSON — WALKER SPACETIME

Mohd Danish Siddiqi

Professor, Department of Mathematics,
Jazan University
msiddiqi@jazanu.edu.sa
Al Maarefah Rd, 82817 Jazan, Saudi Arabia

Shah Alam Siddiqui

Professor, Department of Mathematics,
Jazan University
sasiddiqui@jazanu.edu.sa
Al Maarefah Rd, 82817 Jazan, Saudi Arabia

Abstract. In this research article, we estimate the behavior of an imperfect fluid generalized Robertson — Walker spacetime (GRW) in terms of k-Yamabe soliton with torse-forming vector field. Besides this, we evaluate a specific situation when the potential vector filed ξ is of the form of gradient i.e., $\xi = \mathrm{grad}(\Psi)$, we extract a Laplace — Poisson equation, and Liouville equation from the quasi k-Yamabe soliton equation.

Key words: k-Yamabe soliton, quasi k-Yamabe soliton, imperfect fluid generalized Robertson — Walker spacetime, torse-forming vector field, Einstein manifold.

Introduction

Symmetry is a beautiful property of this universe. It is also one of the fundamental concepts that can describe the laws of nature such as from general relativity to quantum mechanics. In 1915, Albert Einstein introduced the theory of "General Relativity of gravity" (GR). In this theory the gravitational field is the spacetime curvature and its source is energy-momentum tensor. All the equations of modern particle physics (astrophysics, plasma physics, nuclear physics etc.) were modeled after the Einstein equations, the equations that describe the evolution of spacetime curvature. The goal to develop differential geometry and relativistic fluids model in the mathematical language are most efficient for understanding general relativity. The main idea of the general theory of relativity is that the spacetime should be described as a curved manifold. According to J. A. Wheeler, “Matter tells spacetime how to bend and spacetime returns the complement by telling matter how to move”.

The spacetime of general relativity and cosmology can be modeled as a connected 4-dimensional time orientated Lorentzian manifold which is a special subclass of pseudo-Riemannian manifolds with Lorentzian metric g with signature $(-,+,+,+)$ has great importance in general relativity. The geometry of Lorentzian manifold begins with the study of nature of vectors on the manifold. Therefore, Lorentzian manifold becomes most suitable choice for the study of general relativity. Alias et al. [1] introduced the notion of generalized Robertson — Walaker (GRW) spacetime, which is a generalization of the Robertson — Walker (RW) spacetime. A GRW spacetime of dimension n is an n-dimensional Lorentzian manifold M. According to Sanchez [27] the GRW-spacetime have application in inhomogeneous spacetime with an isotropic radiation. O'Neil [26] in his book listed that a RW-spacetime is a imperfect fluid spacetime. For dimension $(n = 4)$ GRW-spacetime is a imperfect fluid if it is RW spacetime.

The symmetries are a part of geometry and thus reveals the physics. There are many symmetries regarding the spacetime geometry and matter. The metric symmetries are important as they simplify solution to many problems. Their main application in general relativity is that they classify solutions of Einstein Field equations. One of these symmetries is solitons associated with the geometric flow of spacetime geometry. Such as Ricci flow and Yamabe flow are important because they can help in understanding the concepts of energy and entropy in general relativity. Ricci soliton and Yamabe soliton are the points at which the curvatures obey a self similarity.

In [2] Ahsan and Ali discussed about the spacetime with Ricci soliton. In [6] Blaga studied the geometrical aspects of a perfect fluid spacetime in terms of Einstein solitons and Ricci solitons. In addition, Venkatesha and Aruna [33] also apply Ricci solitons and study the perfect fluid spacetime with the potential vector field. More recently, Chen and Deshmukh [11] studied more general notion namely *generalized quasi Yamabe solitons*. In [18] Jun and Siddiqi also studied almost quasi-Yamabe solitons on Lorentzian concircular structure manifolds. Moreover, various authors extensively studied about solitons with spacetime in various ways (for more details see [3; 6; 8; 16; 29; 30]). We notice that Chen [13] defined k-almost Yamabe soliton. Motivated by this, we consider k-Yamabe soliton in this study.

Therefore, motivated by of the above researches and remarks, in this paper, we study the geometry of a imperfect fluid spacetime admitting k-Yamabe soliton.

A k-Yamabe soliton is a Riemannian metric g on smooth manifold M such that a smooth vector field U, a soliton constant γ and a funstion $k : M \longrightarrow \mathbb{R}$ satisfy

$$\frac{k}{2}\mathfrak{L}_U g = (R - \gamma)g, \tag{1}$$

where R indicate the scalar curvature of M; $\mathfrak{L}_U$ is the Lie-derivative in the direction of potential vector field U. Represent the k-Yamabe soliton by (g, U, k, γ). The k-Yamabe soliton is said to be expanding, steady or shrinking, according as $\gamma < 0$, $\gamma = 0$ or $\gamma > 0$

respectively.

We call a Riemannian manifold M a quasi k-Yamabe soliton if it admits a vector field U such that [11]

$$\frac{k}{2}\mathfrak{L}_U g = (R-\gamma)g + \omega\eta\otimes\eta, \tag{2}$$

for some constant γ and some function ω, where η is the dual 1-form. We indicate the quasi k-Yamabe soliton satisfying (2) by $(g, U, k, \gamma, \omega)$. For more details about solitons see [4; 5; 10; 11; 31].

Geometry of Ricci — Yamabe solitons, can develop a bridge between a curvature inheritance symmetry of perfect fluid spacetime manifold (semi-Riemannian manifold) and class of k-Yamabe solitons. In support of this relation we construct three mathematical models of semi-Riemannian manifolds with k-Yamabe solitons. As an application to relativity by investigating the kinematic and dynamic nature of spacetime, we present a physical models of three classes namely, shrinking, steady and expanding of perfact fluid solution of k-Yamabe solitons spacetime.

To deal with three special classes of k-Yamabe solitons, namely, shrinking ($\gamma < 0$) which exists on a maximal time interval $-\infty < t < b$ where $b < \infty$, steady ($\gamma = 0$) which exists for all time or expanding ($\gamma > 0$) which exists on maximal time interval $a < t < \infty$, where $a > -\infty$ [14]. These classes yields example of *ancient, eternal and immortal solution*, respectively. Also, solutions of Einstein gravity coupled to a free mass less scalar field with nonzero cosmological constant are associated with shrinking or expanding k-Yamabe solitons.

1. Preliminaries

The energy-momentum tensor plays the major role as a matter content of the spacetime, matter is assumed to be fluid having density, pressure and having dynamical and kinematical quantities like velocity, acceleration, vorticity, shear and expansion [32]. The matter content of the universe is consider to behave like a perfect fluid and imperfect fluid spacetime in standard cosmological models. The Brans-Dicke-like field of scalar-tensor gravity can be described as an imperfect fluid in an approach in which the field equations are regarded as effective Einstein equations. In Einstein's theory, an effective imperfect fluid description can be given for a canonical, Generalized-Robertson — Walker (GRW)-spaces used in Friedmannian cosmology [23].

The stress-energy-momentum tensor T of an imperfect fluid GRW-spacetime in the following form ([25; 26])

$$T(U,V) = pg(U,V) + (\sigma + p)\eta(U)\eta(V) + P(U,V), \tag{3}$$

where σ, p are the energy density and isotropic pressure respectively and P denotes the an isotropic pressure tensor to the viscous fluid (incompressible fluid).

Further, example of energy-momentum tensor are energy-momentum tensor of electromagnetism and scalar field theory.

The field equation governing the perfect fluid motion is Einstein's gravitational equation [26]

$$S(U,V) + \left(\lambda - \frac{R}{2}\right) g(U,V) = \kappa T(U,V), \tag{4}$$

for any $U, V \in \chi(M)$, where λ is the cosmological constant, κ is the gravitational constant (which can be taken $8\pi G$, with G the universal gravitational constant), S is the Ricci tensor and R is the scalar curvature of g. They are obtained from Einstein's equations by adding a cosmological constant in order to get a static universe, according to Einstein's idea. In modern cosmology, it is considered as a candidate for dark energy, the cause of the acceleration of the expansion of the universe.

Also, From equations (3) and (4) we obtain the Einstein's equation for an imperfect fluid GRW-spacetime

$$S(U,V) = \left(-\lambda + \frac{R}{2} + \kappa p\right) g(U,V) + \kappa(\sigma + p)\eta(U)\eta(V) + \kappa P(U,V). \tag{5}$$

2. Imperfect fluid generalized Robertson — Walker spacetime

In this, section, we have discussed the basic ingredients about the GRW spacetime.

Let (M^4, g) be a relativistic imperfect fluid GRW-spacetime satisfying (5). Contracting (5) and assumed that $g(\xi,\xi) = -1$, we obtain

$$R = 4\lambda - \kappa[3p - \sigma + J], \tag{6}$$

where $J = trace(P)$. Therefore

$$S(U,V) = \left(\lambda - \frac{\kappa}{2}[p - \sigma + 2J]\right) g(U,V) + \kappa(\sigma + p)\eta(U)\eta(V) + \kappa P(U,V). \tag{7}$$

$$QU = aU + b\eta(U)\xi, \tag{8}$$

where $a = \left(\lambda - \frac{\kappa}{2}(p - \sigma + 2J)\right)$ and $b = \kappa(1 + \sigma + p)$. Also

$$S(\xi,\xi) = \frac{\kappa}{2}[3p + \sigma + 2(J + I)] - \lambda, \tag{9}$$

where $I = P(\xi,\xi)$.

Definition 1. *A vector field γ_j on a semi-Riemannian manifold is said to be torse-forming vector field if* [35]

$$\nabla_k \gamma_j = \omega_k \gamma_j + \varphi g_{kj}, \tag{10}$$

where φ is a scalar function and ω_k, non-vanishing 1-form.

It is noticed that a unit timelike torse-forming vector field u_i on a semi-Riemannain manifold M takes the following form [35]:

$$\nabla_k u_j = \varphi(g_{kj} + u_k u_j). \tag{11}$$

Motivated by the following results (see [11; 24]) we have the following:

Theorem 1. *A Lorentzian manifold M with $dim(M) \geq 3$ is a generalized Robertson — Walker spacetime (GRW) if and only if it admits a time like concircular vector field.*

In 2017, Mantica and Molinari [24] has discovered the necessary and sufficient conditions for the Lorentzian manifold to be generalized Robertson — Walker spacetime (GRW) if and only if it admits a unit timelike torse-forming vector field $\nabla_k u_j = \varphi(g_{kj} + u_k u_j)$.

Now, follow by the above equations and definition, we have the following theorem for GRW-spacetime in terms of global expressions [9].

Theorem 2. *On an imperfect fluid GRW-spacetime with a unit timelike torse-forming vector filed* ξ*, the following relation hold:*

$$\eta(\nabla_U \xi) = 0, \quad \nabla_\xi \xi = 0, \tag{12}$$

$$(\nabla_U \eta)(V) = \varphi[g(U,V) + \eta(U)\eta(V)], \tag{13}$$

$$R(U,V)\xi = \varphi[\eta(V)U - \eta(U)V], \tag{14}$$

$$R(\xi,U)V = \varphi[\eta(V)U - g(U,V)\xi], \tag{15}$$

$$\eta(R(U,V)W =) = \varphi[\eta(U)g(V,W) - \eta(V)g(Y,W)], \tag{16}$$

$$(\mathcal{L}_\xi g)(U,V) = 2\varphi[g(U,V) + \eta(U)\eta(V)], \tag{17}$$

$$S(U,\xi) = -3\varphi\eta(U). \tag{18}$$

Proof. To compute $(\nabla_U \eta)(V) = U(\eta(V) - \eta(\nabla_U V)) = U(g(V,\xi) - g(\nabla_U V,\xi)) = g(V, \nabla_U \xi) = \varphi[g(U,V) + \eta(U)\eta(V)]$. In particular $(\nabla_\xi \eta)(V) = 0$. The relation (12) can be obtained by (11).

Substituting the expression of $\nabla_U \xi$ from (11) into $R(U,V)\xi = \nabla_U \nabla_V \xi - \nabla_V \nabla_U \xi - \nabla_{[U,V]}\xi$ and by direct calculation we have find the relation (14), (15) and (18).

Now, Lie derivative g along ξ, followed by straight forward computation we get (17).

3. Geometrical characteristics of imperfect fluid GRW-spacetime

In this section, we have discussed the properties of a new curvature tenor called, semi-conformal curvature tensor, and its relationship with imperfect fluid GRW-spacetime.

In 2017, Kim [20] introduced curvature like-tensor field which remain invariant under conharmonic transformation. He named new tensor as *semi-conformal* curvature tensor denoted by P. For a semi-Riemannian manifold M^n with metric g, this tensor is defined as [21]

$$P(U,V)W = -(n-2)\delta C(U,V)W + [\varepsilon + (n-2)\delta]L(U,V)W, \tag{19}$$

provided the constant ε and δ are not simultaneously zero, where C and L are conformal curvature tensor and conharmonic curvature tensor respectively.

Now, an imperfect fluid GRW-spacetime with a unit timelike torse-forming vector filed of dimension 4 is said to be semi-conformally flat, if the semi-conformal curvature tensor P vanishes and is defined by the above equation (19) [20].

Let (M^4, g) be a semi-conformallly flat imperfect fluid GRW-spacetime with a unit timelike torse-forming vector filed ξ. As $P(U,V)W = 0$, we have $\operatorname{div} P = 0$, where div denotes the divergence. Now, from (19) we have

$$(\nabla_U S)(V,W) - (\nabla_V S)(U,W) = \frac{\delta}{3\varepsilon}[U(R)g(V,W) - V(R)g(U,W)], \tag{20}$$

or

$$g((\nabla_U Q)V - (\nabla_V Q)U, W) = \frac{\delta}{3\varepsilon}[U(R)g(V,W) - V(R)g(U,W)]. \tag{21}$$

Here scalar curvature R is constant and from (8) and (21) leads to

$$0 = (\nabla_U Q) - (\nabla_V Q)U = b[(\nabla_U \eta)(V)\xi + \eta(V)\nabla_U \xi - (\nabla_V \eta)(U)\xi - \eta(V)\nabla_U \xi]. \tag{22}$$

Then from (11) and (13), we find that

$$b\left[\eta(V)U - \eta(U)V\right] = 0, \tag{23}$$

which shows that $b = 0$, implies that $p = -(\sigma + 1)$, the energy-momentum tensor is Lorentz-invariant and during this case we can discuss about vacuum.

From (8), we have $QU = aU$. So $P = 0$ implies

$$R(U,V)W = \frac{2\delta(\lambda + \kappa[\frac{1}{2} + \sigma + J])}{3\varepsilon}[g(V,W)U - g(U,W)V], \tag{24}$$

which means (M^4, g) is of constant curvature $\frac{2\delta(\lambda+\kappa[\frac{1}{2}+\sigma+J])}{3\varepsilon}$. This lead the following result:

Theorem 3. *If imperfect fluid GRW-spacetime with a unit timelike torse-forming vector filed* ξ*, is semi-conformally flat, then the stress-energy tensor is Lorentz-invariant and is of constant curvature* $\frac{2\delta(\lambda+\kappa[\frac{1}{2}+\sigma+J])}{3\varepsilon}$.

A pseudo-Riameannian manifold is said to be quasi-constant curvature if the curvature tensor of the type $(0,4)$ satisfies

$$R(U,V,W,W') = m[g(V,W)g(U,W') - g(U,W)g(V,W')] + \tag{25}$$

$$+ n[g(U,W')\eta(V)\eta(W) - g(U,W)\eta(V)\eta(W') +$$

$$+ g(V,W)\eta(U)\eta(W') - g(V,W')\eta(U)\eta(W')],$$

where m and n are scalars and η could be a non-zero 1-form such that $g(U,Z) = \eta(U)$ for all U, Z are unit vector field. The notion of a manifold of quasi-constant curvature was introduced by Yano [34].

Now, from equation (2), we have the following.

Corollary 1. *A semi-conformally flat imperfect fluid GRW-spacetime with a unit timelike torse-forming vector filed* ξ*, is of quasi-constant curvature with* $m = \frac{2\delta(\lambda+\kappa[\frac{1}{2}+\sigma+J])}{3\varepsilon}$ *and* $n = 0$ *in* (25).

We know that manifold of constant curvature is Einstein manifold. From Theorem (3), we state the following theorem.

Theorem 4. *A semi-conformally flat imperfect fluid GRW-spacetime with a unit timelike torse-forming vector filed* ξ*, is an Einstein.*

A pseudo-Riemannian manifold (M, g) is said to be *semi-symmetric* and *Ricci semi-symmetric* if (M, g) holds the condition $R(U,V) \cdot R = 0$ and $R(U,V) \cdot S = 0$, respectively. The condition $R(U,V) \cdot R = 0$ implies $R(U,V) \cdot S = 0$, but converse need not to be true.

Now, we prove the following;

Theorem 5. *A semi-conformally flat imperfect fluid GRW-spacetime with a unit timelike torse-forming vector filed* ξ*, is semi-symmetric and Ricci semi-symmetric.*

Proof. Using, equation (2) we can easily show that $R(U,V) \cdot R = 0$ and this condition implies $R(U,V) \cdot S = 0$.

4. *k*-Yamabe soliton on imperfect fluid GRW-spacetime

This section deal with the study of k-Yamabe soliton in an imperfect fluid GRW-spacetime whose unit timelike velocity vector filed ξ is torse-forming.

Now, taking $V = \xi$, equation (1) becomes

$$\frac{k}{2}\mathfrak{L}_U g(E,F) = (R-\gamma)g(E,F), \tag{26}$$

where R is scalar curvature. Now, using (17), we find

$$(k\varphi - R - \gamma)g(E,F) + k\varphi\eta(U)\eta(V) = 0. \tag{27}$$

Putting $E = F = \xi$ in (27) and using (9), we get

$$\gamma = 4\lambda - \kappa(3p - \sigma + J). \tag{28}$$

Also, using the above equation in (26), we have

$$\mathfrak{L}_\xi g = 0. \tag{29}$$

Thus, ξ is a Killing vector field. Hence we have the following conclusions:

Theorem 6. *If an imperfect fluid GRW-spacetime with a unit timelike torse-forming vector filed ξ admits a k-Yamabe soliton (g,ξ,k,γ), ξ being the Reeb vector field, then the scalar curvature R is constant and ξ becomes a Killing vector field.*

Theorem 7. *If an imperfect fluid GRW-spacetime with a unit timelike torse-forming vector filed ξ admits a k-Yamabe soliton (g,ξ,k,γ) then k Yamabe solitons is expanding, steady and shrinking according as*

1) $\lambda < \frac{\kappa(3p-\sigma+J)}{4}$,

2) $\lambda = \frac{\kappa(3p-\sigma+J)}{4}$ *and*

3) $\lambda > \frac{\kappa(3p-\sigma+J)}{4}$ *respectively.*

Moreover, in case of a perfect fluid GRW-spacetime $J = 0$, then we turn up the following

Corollary 2. *If a perfect fluid GRW-spacetime with a unit timelike torse-forming vector filed ξ admits k-Yamabe soliton (g,ξ,k,γ) then k-Yamabe solitons is expanding, steady and shrinking according as*

1) $\lambda < \frac{\kappa(3p-\sigma)}{4}$,

2) $\lambda = \frac{\kappa(3p-\sigma)}{4}$ *and*

3) $\lambda > \frac{\kappa(3p-\sigma)}{4}$ *respectively.*

5. Quasi *k*-Yamabe soliton on imperfect fluid GRW-spacetime

In this segment, we deal with quasi k-Yamabe soliton on an imperfect fluid GRW-spacetime if the potential vector field ξ of the quasi k-Yamabe soliton is of the form, i.e., $\xi = \mathrm{grad}(\Psi)$.

Let us assume the equation

$$k\mathfrak{L}_{\xi}g(E,F) + 2(\gamma - R)g(E,F) - 2\omega\eta(E)\eta(F), \tag{30}$$

where g is a Riemannian metric, R is scalar curvature, ξ is a vector field, η is a 1 form and γ is real constant, and ω is some function such that $\mu : M \longrightarrow \mathbb{R}$. The data $(g, \xi, k, \gamma, \omega)$ satisfies the equation (30) is said to be *quasi-k-Yamabe soliton*. In particular, if $\omega = 0$, (g, ξ, k, γ) is a Yamabe soliton.
From (30) we turn up

$$(\gamma - R)g(E,F) = -\omega\eta(E)\eta(F) - \frac{k}{2}[g(\nabla_F \xi, E) + g(F, \nabla_E \xi)], \tag{31}$$

for any $E, F, \in \chi(M)$.
Apply contraction on (31) we get

$$4\gamma - \omega = 4R - k\,\mathrm{div}(\xi). \tag{32}$$

Now putting $E = F = \xi$ in (31), we obtain

$$\omega - \gamma = -R. \tag{33}$$

Therefore

$$\begin{cases} \gamma = R - \frac{k}{3}\,\mathrm{div}(\xi) \\ \omega = -2R - \frac{k}{3}\,\mathrm{div}(\xi). \end{cases} \tag{34}$$

Using (34) we can conclude the followings.
Theorem 8. *Let (M^4, g) be an imperfect fluid GRW-spacetime and η be the g-dual 1-form of the gradient vector field $\xi = \mathrm{grad}(\Psi)$. If (30) determines a quasi k-Yamabe soliton, then the Laplace — Poisson equation satisfying by Ψ becomes*

$$\Delta(\Psi) = \frac{3}{k}[\omega + 2R]. \tag{35}$$

Theorem 9. *Let $(g, \xi, k, \gamma, \omega)$ be quasi k-Yamabe soliton in an an imperfect fluid GRW-spacetime. Then the soliton is steady, expanding and shrinking according as*

1) $\lambda < \frac{3\kappa}{4}\left\{(3p - \sigma + J) + \frac{k}{12}\,\mathrm{div}\,\xi\right\}$,

2) $\lambda = \frac{3\kappa}{4}\left\{(3p - \sigma + J) + \frac{k}{12}\,\mathrm{div}\,\xi\right\}$ *and*

3) $\lambda > \frac{3\kappa}{4}\left\{(3p - \sigma + J) + \frac{k}{12}\,\mathrm{div}\,\xi\right\}$ *respectively.*

Corollary 3. *Let $(g, \xi, k, \gamma, \omega)$ be quasi k-Yamabe soliton in an an imperfect fluid GRW-spacetime. Then the soliton is steady, expanding and shrinking according as*

1) $\lambda < \frac{3\kappa}{4}\left\{(3p-\sigma)+\frac{k}{12}\operatorname{div}\xi\right\}$,

2) $\lambda = \frac{3\kappa}{4}\left\{(3p-\sigma)+\frac{k}{12}\operatorname{div}\xi\right\}$ *and*

3) $\lambda > \frac{3\kappa}{4}\left\{(3p-\sigma)+\frac{k}{12}\operatorname{div}\xi\right\}$ *respectively.*

Remark. It is noted that a function $f : M \longrightarrow \mathbb{R}$ is said to be harmonic if $\Delta f = 0$, where Δ is the Poisson — Laplace operator on M [36].

Again, from (34) we turn up

$$div(\xi) = \frac{3}{k}(R-\gamma). \tag{36}$$

Since $k \neq 0$ and $\xi = \operatorname{grad}(\Psi)$ type, so we get $\gamma = R = \text{const}$, which implies

$$\Delta\Psi = 0. \tag{37}$$

Thus using the above equation and remark (5) we can conclude the following

Theorem 10. *If a (M^4, g) be an imperfect fluid GRW-spacetime with a gradient vector field of type $\xi = \operatorname{grad}(\Psi)$ admits quasi k-Yamabe soliton, then the function Ψ is a harmonic if and only if the value of γ is equal to the scalar curvature R.*

Corollary 4. *If a (M^4, g) be an imperfect fluid GRW-spacetime with a gradient vector field of type $\xi = \operatorname{grad}(\Psi)$ admits a k-Yamabe soliton, then the function Ψ is a harmonic if and only if the scalar curvature $R = \gamma$.*

Therefore we infer the above remark and Theorem 10) we entails the following

Theorem 11. *If a (M^4, g) be an imperfect fluid GRW-spacetime with a gradient vector field of type $\xi = \operatorname{grad}(\Psi)$ admits a quasi k-Yamabe soliton and Ψ is a harmonic function, then the quasi k-Yamabe soliton is expanding.*

Corollary 5. *If a (M^4, g) be an imperfect fluid GRW-spacetime with a gradient vector field of type $\xi = grad(\Psi)$ admits a k-Yamabe soliton and Ψ is a harmonic function, then the k-Yamabe soliton is expanding.*

6. Physical model of Laplace — Poisson equation

The general theory of solution of Laplace — Poisson equation is known as *potential theory* and the solution of Laplacez — Poisson equation are harmonic functions, which are important in branches of physics, electrostatics, gravitation and fluid dynamics. In modern physics, there are two fundamental forces of the nature known at the time, namely, gravity and the electrostatics forces, could be modeled using functions called the gravitational potential and electrostatics potential both of which satisfy Laplace equation.

Example 1. Let us assume the physical phenomena, if Ψ be the gravitational filed, ρ the mass density and G the gravitational constant. The Gauss's law of gravitational in differential form is

$$\nabla\Psi = -4\pi G\rho. \tag{38}$$

In case of gravitational field, Ψ is conservative and can be expressed as the negative gradient of gravitational potential, i.e., $\Psi = -gradf$ then by the Gauss's law of gravitational, we have

$$\nabla^2 f = 4\pi G\rho. \tag{39}$$

This physical phenomena is directly identical to the Theorem (8) and equation (35), which is a Laplace — Poisson equation with potential vector filed of gradient type i.e $\xi = grad(\Psi)$.

Remark. Also, for $\psi \in C^\infty(M)$ and the vector field ξ a straight forward calculation gives

$$\mathrm{div}(\psi\xi) = \xi(d\psi) + \psi\,\mathrm{div}\,\xi. \tag{40}$$

The function $\psi \in C^\infty(M)$ is a last multiplier of vector field ξ with respect to g if $\mathrm{div}(\psi\xi) = 0$. The corresponding equation

$$\xi(d\ln\psi) = -\,\mathrm{div}(\xi) \tag{41}$$

is called the **Liouville equation** of the vector field ξ with respect to g (for more details see [19]).

Now, infer the above remark and equation (34), we obtain the following result.

Theorem 12. *Let (M^4, g) be an imperfect fluid GRW-spacetime and η be the g-dual 1-form of the gradient vector field $\xi = \mathrm{grad}(\Psi)$, $\Psi \in C^\infty(M)$. If* (30) *defines a quasi k-Yamabe soliton, then the Liouville equation satisfying by ψ and ξ becomes*

$$\xi(d\ln\psi) = \frac{3}{k}(\gamma - R). \tag{42}$$

7. Significance of Liouville equation in Physics

The Liouville equation, describe the nature of incompressible fluid in phase space. It explain the evolution of an ensemble or collection of classical system in phase space. Liouville equation describe the flow of whole distribution, the motion analogous to a dye in an incompressible fluid, whereas the motion of a system of the ensemble is given by Hamiltonian equation. Moreover, symmetry is invariance under time translation, and the generator of symmetry is Hamiltonian. In fact the phase points of ensemble are neither created nor destroyed. In addition Liouville equation is a persistent for the flux and phase space density σ assumed as

$$\frac{\partial\sigma}{\partial t} = -\nabla . J^{\longrightarrow}. \tag{43}$$

REFERENCES

1. Alias L., Romero A., Sanchez M. Uniqueness of Complete Spacelike Hypersurfaces of Constant Mean Curvature in Generalized Robertson — Walker Spacetimes. *Gen. Relativ. Gravit*, 1995, vol. 27, iss. 1, pp. 71-84. DOI: https://doi.org/10.1007/BF02105675.

2. Ali M., Ahsan Z. Ricci Solitons and Symmetries of Space Time Manifold of General Relativity. *Journal of Advanced Research on Classical and Modern Geometries*, 2014, vol. 1, iss. 2, pp. 75-84.

3. Ahsan Z., Siddiqui S.A. Concircular Curvature Tensor and Fluid Spacetimes. *Int. J. Theor Phys*, 2009, vol. 48, iss. 11, pp. 3202-3212. DOI: https://doi.org/10.1007/s10773-009-0121-z.

4. Blaga A.M. Harmonic Aspects in an η-Ricci Soliton. *Int. Elect. J. Geom.*, 2020, vol. 13, iss. 1, pp. 41-49.

5. Blaga A.M. On Gradient η-Einstein Solitons. *Kragujevac J. Math*, 2018, vol. 42, iss. 2, pp. 229-237.

6. Blaga A.M. Solitons and Geometrical Structures in a Perfect Fluid Spacetime. *Rocky Mountain J. Math*, 2020, vol. 50, no. 1, pp. 41-53. DOI: https://doi.org/10.48550/arXiv.1705.04094.

7. Catino G., Mazzieri L. Gradient Einstein Solitons. *Nonlinear Anal*, 2016, vol. 132, pp. 66-94.

8. Chaki M.C., Ray S. Space-Times with Covariant-Constant Energy-Momentum Tensor. *Int. J. Theor Phys*, 1996, vol. 35, iss. 5, pp. 1027-1032. DOI: https://doi.org/10.1007/BF02302387.

9. Chaubey S.K. Generalized Robertson — Walker Space-Times with W_1-Curvature Tensor. *J. Phys Math.*, 2019, vol. 10, iss. 2, article ID: 1000303.

10. Chen B.-Y. A Simple Characterization of Generalized Robertson — Walker Spacetime. *Gen. Relativity Gravit*, 2014, vol. 46, article ID: 1833. DOI: https://doi.org/10.1007/s10714-014-1833-9.

11. Chen B.Y., Desahmukh S. Yamabe and Quasi-Yamabe Soliton on Euclidean Submanifolds. *Mediterranean Journal of Mathematics*, 2018, vol. 15, article ID: 194. DOI: https://doi.org/10.1007/s00009-018-1237-2.

12. Crasmareanu M. Last Multipliers on Weighted Manifolds and the Weighted Liouville Equation. *UPB Scientific Bulletin, Series A: Applied Mathematics and Physics*, 2015, vol. 77, iss. 3, pp. 53-58.

13. Cui X., Chen X. The k-Almost Yamabe Solitons and Contact Metric Manifolds. *Rocky Mountain J. Math.*, 2021, vol. 51, iss. 1, pp. 125-137. DOI: https://doi.org/10.1216/rmj.2021.51.125.

14. Duggal K. Almost Ricci Solitons and Physical Applications. *International Electronic Journal of Geometry V*, 2017, vol. 10, iss. 2, pp. 1-10.

15. De U.C., Ghosh G.C. On Quasi Einstein Manifolds. *Period. Math. Hungar*, 2004, vol. 48, iss. 12, pp. 223-231. DOI: https://doi.org/10.1023/B:MAHU.0000038977.94711.ab.

16. De U.C., Velimirovic L. Spacetimes with Semisymmetric Energy-Momentum Tensor. *Int. J. Theor Phys.*, 2015, vol. 54, pp. 1779-1783. DOI: https://doi.org/10.1007/s10773-014-2381-5.

17. Hamilton R.S. The Ricci Flow on Surfaces. *Mathematics and General Relativity*. Santa Cruz, Amer. Math. Soc., 1988, vol. 71, pp. 237-262.

18. Jun J.B., Siddiqi M.D. Almost Quasi-Yamabe Solitons on Lorentzian Concircular Structure Manifolds-$[(LCS)_n]$. *Honam Mathematical Journal*, 2020, vol. 42, iss. 3, pp. 521-536.

19. Katz M. Liouville's Equation for Curvature and Systolic Defect. *arXiv:1105.0553v2[math.DG]*. URL: https://arxiv.org/abs/1105.0553.

20. Kim J. A Type of Conformal Curvature Tensor. *Far East J. Math. Sci.*, 2016, vol. 99, iss. 1, pp. 61-74. DOI: https://doi.org/10.17654/MS099010061.

21. Kim J. On Pseudo Semiconformally Symmetric Manifolds. *Bull. Korean Math. Soc.*, 2017, vol. 54, iss. 1, pp. 177-186. DOI: https://doi.org/10.4134/BKMS.b151007.

22. Mantica C.A., Molinari L.G., De U.C. A Condition for a Perfect Fluid Spacetime to be a Generalized Robertson — Walker Spacetimes, a Survey. *J. Math. Phys.*, 2016, vol. 57, no. 2, article ID: 049901. DOI: https://doi.org/10.1063/1.4941942.

23. Mantica C.A., Suh Y.J, De U.C. A Note on Generalized Robertson — Walker Spacetimes. *Int. J. Geom. Math. Mod. Phys.*, 2016, vol. 13, no. 6, article ID: 1650079. DOI: https://doi.org/10.1142/S0219887816500791.

24. Mantica C.A., Molinari L.G. Generalized Robertson — Walker Spacetimes — A

Survey. *Int. J. Geom. Math. Mod. Phys.*, 2017, vol. 14, no. 3, article ID: 102502. DOI: https://doi.org/10.1142/S021988781730001X.

25. Novello M., Reboucas M.J. The Stability of a Rotating Universe. *Astrophys. J.*, 1978, vol. 225, pp. 719-724.

26. O'Neill B. *Semi-Riemannian Geometry with Applications to Relativity*. New York, Academic Press, 1983. xii, 468 p.

27. Sanchez M. On the Geometry of Generalized Robertson — Walker Spacetime Geodesics. *Gen. Relativ. Gravit.*, 1998, vol. 27, pp. 71-84.

28. Shaikh A.A., Jana S.K. A Pseudo-Quasi-Conformal Curvature Tensor on a Riemannian Manifold. *South East Asian J. Math. Math. Sci.*, 2005, vol. 4, iss. 1, pp. 15-20.

29. Siddiqi M.D., Siddiqui S.A. Conformal Ricci Soliton and Geometrical Structure in a Perfect Fluid Spacetime. *Int. J. Geom. Methods Mod. Phys.*, 2020, vol. 17, iss. 6, article ID: 2050083. DOI: https://doi.org/10.1142/S0219887820500838.

30. Siddiqi M.D. Generalized Yamabe Solitons on Trans-Sasakian Manifolds. *Bulletin of Institute of Mathematics*, 2020, iss. 3, pp. 77-85.

31. Siddiqi M.D. Ricci ρ-Soliton and Geometrical Structure in a Dust Fluid and Viscous Fluid Spacetime. *Bulg. J. Phys.*, 2019, vol. 46, pp. 163-173.

32. Stephani H. *General Relativity-An Introduction to the Theory of Gravitational Field*. Cambridge, Cambridge University Press, 1982. 314 p.

33. Venkatesha V., Kumara H.A. Ricci Solitons and Geometrical Structure in a Perfect Fluid Spacetime with Torse-Forming Vector Filed. *Afrika Mathematika*, 2019, vol. 30, pp. 725-736. DOI: https://doi.org/10.1007/s13370-019-00679-y.

34. Yano K. *Integral Formulas in Riemannian Geometry*. New York, Marcel Dekker, 1970. 466 p. DOI: https://doi.org/10.1017/S0008439500031520.

35. Yano K. On Torse Forming Direction in a Riemannian Space. *Proc. Imp. Acad. Tokyo*, 1994, vol. 20, pp. 340-345.

36. Yau S.T. Harmonic Functions on Complete Riemannian Manifolds. *Commu. Pure. Appl. Math*, 1975, vol. 28, pp. 201-228.

СОЛИТОНЫ k-ЯМАБЕ И КВАЗИ k-ЯМАБЕ НА ОБОБЩЕННОМ НЕСОВЕРШЕННОМ ЖИДКОМ ПРОСТРАНСТВЕ-ВРЕМЕНИ РОБЕРТСОНА — УОЛКЕРА

Мод Даниш Сиддики

Профессор кафедры математики,
Джазанский университет
msiddiqi@jazanu.edu.sa
проезд Эль Маарефа, 82817 г. Джазан, Саудовская Аравия

Шах Алам Сиддики

Профессор кафедры математики,
Джазанский университет
sasiddiqui@jazanu.edu.sa
проезд Эль Маарефа, 82817 г. Джазан, Саудовская Аравия

Аннотация. В статье оценивается поведение несовершенного жидкого обобщенного пространства-времени Робертсона — Уолкера (GRW) в терминах солитона k-Ямабе с торсообразующим векторным полем. Кроме того, рассматривается частный случай, когда потенциальный вектор ξ имеет форму градиента, то есть $\xi = \mathrm{grad}(\Psi)$. Нами получено уравнение Лапласа — Пуассона и уравнение Лиувилля из уравнения квази k-Ямабе солитона.

Ключевые слова: солитон k-Ямабе, солитон квази k-Ямабе, несовершенное жидкое обобщенное пространство-время Робертсона — Уолкера, торсообразующее векторное поле, многообразие Эйнштейна.