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UNIQUENESS THEOREM OF RECONSTRUCTION OF PREIMAGE BY ITS IMAGE UNDER DEGENERATE MAPPING[1]

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Abstract. One of the urgent problems in the construction of computer vision systems is the problem of determining the spatial orientation and spatial position of an object from a photograph. For example, this task is especially important for autonomous driving systems, where the positioning of nearby vehicles is a key issue for autonomous vehicles in an urban environment. In this regard, for example, in 2019, the Baidu Robotics and Autonomous Driving Laboratory together with Peking University, set an appropriate task for the Kaggle community (https://www.kaggle.com/c/pku-autonomous-driving) and provided more than 60,000 copies of three-dimensional cars marked from 5,277 real images. In this article, we formulate some results that are the mathematical basis for substantiating methods for solving the above-mentioned reconstruction problems in computer vision systems.

Key words: central projection, degenerate maps, preimage restoration, transformation groups, computer vision systems.

Introduction

We choose the finite subset $P \subset \mathbb{R}^3$ as a 3D model of objects in space. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be central projection which is defined by formulas

$$\pi(x, y, z) = (x/z, y/z), \quad z \neq 0.$$

It is clear that the set P can't be reconstructed by its image $P' = \pi(P)$. But if we introduce some restrictions on the structure of finite set P we can hope that this reconstruction will be possible. Note, that earlier we considered some problems which are closed to the considered here:

- the problem of determination the plane of the circle in $\mathbb{R}^3$ by its projection [1];
- the problem of determination the spatial triangle with defined angles by plane triangle of its projection [6];
- the problem of the surface reconstruction for objects in space by its image [2];
- the problem of determination the profile function of the visible part of the rotation surface by its projection [5].

In addition, we developed machine learning models using artificial Convolutional Neural Network (CNN) and performed a series of experiments for determination of the spatial orientations vehicles on the photographs [3; 4].

The goal of this article to give mathematical foundations for explanation of possibility the solving of the reconstruction problem. The key statement to solve this problem is uniqueness theorem. In other words, we propose that structure of the set P is described by the admissible transformations $F : \mathbb{R}^3 \to \mathbb{R}^3$ of this set. Uniqueness theorem presuppose that if $\pi(P) = \pi(F(P))$ then $P = F(P)$.

Now we give strict mathematical formulations. Let $P \subset \mathbb{R}^3$ be some finite set. Consider some family Φ of differentiable transformations of $\mathbb{R}^3$. What the conditions we set for this family which imply the fact: the equality $\pi(P) = \pi(F(P'))$ for some transformation $F : \mathbb{R}^3 \to \mathbb{R}^3$, $F \in \Phi$ leads to $P' = P$. We shall consider general case of degenerate mappings.

Let $f : \mathbb{R}^n \to \mathbb{R}^m$, $1 < m < n$ be some differentiable mapping. We consider Lie group $\mathbb{G}$ with unit $e \in \mathbb{G}$ of differentiable transformations of $\mathbb{R}^n$ and let $A_{\mathbb{G}}$ be its Lie algebra. It is well known that for each $X \in A_{\mathbb{G}}$ it corresponds vector field X^* in $\mathbb{R}^n$ such that $exp(tX)$ be local 1-parameter of local transformations of the vector field X^* (see, for example, Proposition 4.1 in [7]). We define the following set for each $X \in A_{\mathbb{G}}$

$$M_X = \{x \in \mathbb{R}^n : X^*(x) \in \text{Ker } df\}.$$

Example. Let $\mathbb{G}$ be a group of parallel transformations in $\mathbb{R}^n$. Then $A_{\mathbb{G}}$ is isomorphic to $\mathbb{R}^n$. Then if $X \in A_{\mathbb{G}}$, then X^* is constant vector field $X^*(x) = X$.

Hausdorff distance is defined by the formula

$$d_H(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} |a - b|, \sup_{b \in B} \inf_{a \in A} |a - b|\}.$$

1. Uniqueness theorem

Consider some finite set $P \subset \mathbb{R}^n$ such that $\forall X \in A_{\mathbb{G}}$ the set P is not subset of M_X.

Theorem 1. *For all $g_0 \in \mathbb{G}$ there is some neighborhood $U \subset \mathbb{G}$ of g_0 such that if for some $g \in U$ it holds*

$$f(g(P)) = f(g_0(P))$$

then $g = g_0$.

For the proof of theorem 1 we use the following result.

Lemma 1. *Let $d_H(A, B)$ denotes Hausdorff distance between subsets $A, B \subset \mathbb{R}^m$. Consider curve $\gamma_t = \exp(tX)$ for $X \in A_{\mathbb{G}}$. We put*

$$h(t) = d_H(f(P), f(\gamma_t(P))).$$

Then

$$\left| \frac{dh(t)}{dt}\Big|_{t=0} \right| = \max_{p\in P} |df(X^*(p))| .$$

Proof. We note that there exists point $p \in P$ such that for all sufficient small $t > 0$ we have

$$d_H(f(P), f(\gamma_t(P))) = |f(p) - f(\gamma_t(p))|.$$

It follows from the fact that the nearest point in the set $f(\gamma_t(P)$ for $p \in P$ is the point $f(\gamma_t(p))$. We set

$$q(t) = f(p) - f(\gamma_t(p)), \quad \mu(t) = |q(t)|$$

It is clear that $q(0) = 0$ and

$$q'(0) = \lim_{t\to 0} \frac{q(t)}{t} = -df\left(\frac{\gamma_t(p)}{dt}\Big|_{t=0}\right) = -df(X^*).$$

Therefore

$$\mu'(0) = \lim_{t\to 0} \frac{\langle q(t), q'(t)\rangle}{\mu(t)} = \frac{|q'(0)|^2}{\mu'(0)},$$

and finitely $\mu'(0) = |q'(0)| = |df(X^*)|$.

Proof of Theorem 1. It is obviously that it is sufficient to prove theorem for $g_0 = e$. We put

$$h_X(t) = d_H(f(P), f(\gamma_t(P))).$$

From lemma 1 it follows that for all $X \in A_{\mathbb{G}}$ there exists $t_X > 0$ such that $h'_X(t) > 0$ for $t \in [0, t_X]$. We define

$$U = \{g \in \mathbb{G} : g = \exp(tX),\ t \in [0, t_X]\}.$$

Now we note that if $g = \exp(sX) \in U$ for some $X \in A_{\mathbb{G}}$ and $s \in [0, t_X]$ such that $f(g(P)) = f(P)$ then $h_X(0) = h_X(s) = 0$. Therefore $h'_X(t^*) = 0$ for some $t^* \in [0, t_X]$. It contradicts to definition of U. Theorem is proved.

Corollary 1. *Let $M \subset \mathbb{R}^3$ some polyhedron and $P = V(M)$ is the set of its vertices. Consider some rotation $R : \mathbb{R}^3 \to \mathbb{R}^3$ by sufficiently small angle $\alpha \geq 0$. If $\pi \circ R(P) = \pi(P)$ then $\alpha = 0$.*

2. Stability Theorem

In addition to the uniqueness theorem, to substantiate the possibility of reconstructing objects, we formulate and prove a stability theorem. This theorem assumes that with a small deformation of the projection of the object, the result of the reconstruction also undergoes small deformation. Let us move on to the exact wording.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be some smooth mapping such that

$$\det df > 0$$

everywhere. We fix some positive $\Delta > 0$. Let $D = df \cdot df^T$ and $0 < \lambda_1 \leq ... \leq \lambda_n$, $E_1, ..., E_n$ eigenvalues and eigenvectors of matrix D. Here $(\cdot)^T$ denotes a transposed matrix of df. We suppose that it exists $1 \leq k \leq n$ such that $\Delta \leq \lambda_{k+1} \leq ... \leq \lambda_n$. We denote by L_k – subspace spanned by vectors $E_1, ..., E_k$.

For each $X \in A_{\mathbb{G}}$ and for $0 \leq \theta \leq \pi/2$ we define the following set

$$M_X(\theta) = \{x \in \mathbb{R}^n : \text{ value of angle } \angle(X^*, L_k) < \theta\}.$$

For $X \in A_{\mathbb{G}}$ we put $g = \exp(X)$ and $\gamma_t = \exp(tX)$ $t \in [0,1]$. Let $\Gamma(p) = \{\gamma_t(p), t \in \in [0,1]\}$ be integral curve of the vector field X^* which joins the points p and $g(p)$. We define the following values

$$L(g, P) = \max_{p \in P} |\Gamma(p)|, \quad l(g, P) = \max_{p \in P} |f(\Gamma(p))|,$$

where $|\Gamma(p)|$ and $|f(\Gamma(p))|$ denote the length of the curves.

Theorem 2. *Let* $X \in A_{\mathbb{G}}$, $g = \exp(X)$. *We suppose that for some* $0 < \theta \leq \pi/2$ *and for all* $t \in [0,1]$ *it holds*

$$\gamma_t(P) \cap M_X(\theta) = \emptyset.$$

Then

$$L(g, P) \leq \frac{l(g, P)}{\sqrt{\Delta} \sin \theta}.$$

Lemma 2. *If* $x \notin M_X(\theta)$ *then*

$$|df(X^*)| \geq |X^*| \sqrt{\Delta} \sin \theta.$$

Proof. Indeed, we have

$$|df(X^*)|^2 = \langle df(X^*), df(X^*) \rangle = \langle df^T \cdot df(X^*), X^* \rangle = \langle D(X^*), X^* \rangle.$$

Let $\alpha_i, i = 1, 2, ..., n$ be the angles between vector X^* and eigenvectors E_i. Then

$$\langle D(X^*), X^* \rangle = |X^*|^2 \sum_{i=1}^{n} \lambda_i \cos^2 \alpha_i.$$

We denote by α the angle between vector X^* and subspace L_k. It is clear that

$$\cos^2 \alpha = \sum_{i=1}^{k} \cos^2 \alpha_i.$$

Therefore

$$\langle D(X^*), X^*\rangle = |X^*|^2 \left(\sum_{i=1}^{k} \lambda_i \cos^2 \alpha_i + \sum_{i=k+1}^{n} \lambda_i \cos^2 \alpha_i\right) \geq$$

$$\geq |X^*|^2 \left(\sum_{i=1}^{k} (\lambda_i - \Delta) \cos^2 \alpha_i + \Delta\right) \geq$$

$$\geq |X^*|^2 \left((\lambda_1 - \Delta) \cos^2 \alpha + \Delta\right) \geq |X^*|^2 \Delta \sin^2 \alpha \geq |X^*|^2 \Delta \sin^2 \theta.$$

Proof of Theorem 2. Let $p \in P$ be point such that

$$|\Gamma(p)| = L(g, P).$$

Using Lemma 2 we conclude

$$l(g, P) \geq |f(\Gamma(p))| = \int_0^1 \left|\frac{df(\gamma_t(p))}{dt}\right| dt =$$

$$\int_0^1 |df(\gamma_t'(p))|\, dt \geq \sqrt{\Delta} \sin\theta \int_0^1 |\gamma_t'(p)| dt = \sqrt{\Delta} \sin\theta |f(\Gamma(p))| = \sqrt{\Delta} \sin\theta L(g, P).$$

NOTE

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REFERENCES

1. Klyachin V.A., Grigoryeva E.G. Algoritm avtomaticheskogo opredeleniya parametrov orientatsii kamery v prostranstve na osnove kharakternykh elementov fotosnimka [Algorithm for Automatic Determination of Camera Orientation Parameters in Space Based on the Characteristic Elements of the Photograph]. *Tendentsii razvitiya nauki i obrazovaniya*, 2018, vol. 45, no. 6, pp. 10-20. DOI: https://doi.org/10.18411/lj-12-2018-125.

2. Klyachin A.A., Klyachin V.A. Algoritm vosstanovleniya poverkhnosti obyekta po ego izobrazheniyu [Algorithm for Restoring the Object Surface From Its Image]. *Matematicheskaya fizika i kompyuternoe modelirovanie* [Mathematical Physics and Computer Simulation], 2021, vol. 24, no. 1, pp. 16-24. DOI: https://doi.org/10.15688/mpcm.jvolsu.2021.1.2.

3. Gordeev A.Y., Klyachin V.A., Kurbanov E.R., Driaba A.Y. Autonomous Mobile Robot with AI Based on Jetson Nano. *Proceedings of the Future Technologies Conference (FTC-2020)*. Cham, Springer, 2021, vol. 1, pp. 190-204. DOI: https://doi.org/10.1007/978-3-030-63128-4_15.

4. Gordeev A.Y., Klyachin V.A. Determination of the Spatial Position of Cars on the Road Using Data from a Camera or DVR. *"Smart Technologies" for Society, State and Economy. ISC 2020. Lecture Notes in Networks and Systems*. Cham, Springer, 2020, vol. 155, pp. 172-180. DOI: https://doi.org/10.1007/978-3-030-59126-7_20.

5. Klyachin V.A., Grigorieva E.G. A 3D Reconstruction Algorithm of a Surface of Revolution from Its Projection. *Journal of Applied and Industrial Mathematics*, 2020, vol. 14, no. 1, pp. 85-91.

6. Klyachin A.A., Klyachin V.A. Existence and Uniqueness Theorems for Solutions of Inverse Problems of Projective Geometry for 3D Reconstruction from Photographs. *Chebyshevskii sbornik*, 2020, vol. 21, no. 4, pp. 117-128.

7. Kobayashi S., Nomizu K. *Foundations of Differential Geometry*. New York, London, John Wiles and Sons, Inc., 1963. 348 p.

ТЕОРЕМЫ ЕДИНСТВЕННОСТИ ВОССТАНОВЛЕНИЯ ПРООБРАЗА ПРИ ВЫРОЖДЕННОМ ПРЕОБРАЗОВАНИИ

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Аннотация. Одной из актуальных проблем построения систем компьютерного зрения является задача определения пространственной ориентации и пространственного положения объекта по фотографии. Например, эта задача особенно важна для автономных систем вождения, где позиционирование ближайших транспортных средств является ключевой проблемой для автономных транспортных средств в городской среде. В связи с этим, например, в 2019 г. Лаборатория робототехники и автономного вождения Baidu совместно с Пекинским университетом поставили перед сообществом Kaggle соответствующую задачу (https://www.kaggle.com/c/pku-autonomous-driving) и предоставили более 60 000 копий трехмерных автомобилей, размеченных с 5 277 реальных изображений. В данной статье мы формулируем некоторые результаты, являющиеся математической основой для обоснования методов решения упомянутых выше задач реконструкции в системах компьютерного зрения.

Ключевые слова: центральная проекция, вырожденные отображения, восстановление прообраза, группы преобразований, системы компьютерного зрения.