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CAN ONE OBSERVE THE BOTTLENECKNESS OF A SPACE BY THE HEAT DISTRIBUTION?[1]

Satoshi Ishiwata

Professor, Department of Mathematical Sciences,
Yamagata University
ishiwata@sci.kj.yamagata-u.ac.jp
Yamagata 990-8560, Japan

Abstract. In this paper we discuss a bottleneck structure of a non-compact manifold appearing in the behavior of the heat kernel. This is regarded as an inverse problem of heat kernel estimates on manifolds with ends obtained in [10] and [8]. As a result, if a non-parabolic manifold is divided into two domains by a partition and we have suitable heat kernel estimates between different domains, we obtain an upper bound of the capacity growth of δ-*skin* of the partition. By this estimate of the capacity, we obtain an upper bound of the first non-zero Neumann eigenvalue of Laplace — Beltrami operator on balls. Under the assumption of an isoperimetric inequality, an upper bound of the volume growth of the δ-skin of the partition is also obtained.

Key words: heat kernel, manifold with ends, inverse problem.

1. Introduction

In the real world, waves, like as electromagnetic waves (light, radiation ray, infrared ray etc...) are good tools to observe a structure of given space (ex. location of obstacles, airplanes, planets etc...). However, in the following situation, heat distribution might be more useful to observe the structure of the space. Consider a space separated into two domains by a "dark" partition (difficult to see by the light). Here, assume that we know the heat distribution well on each separated domain and we have some data of the heat distribution between two domains. Then what can one observe the dark partition? Can one see how large it is? Can one see inside of the partition?

This problem is inspired by a recent progress of the study of the heat kernel on non-compact Riemannian manifolds. Let M be a geodesically complete Riemannian manifold. The heat kernel $p(t,x,y)$ is the minimal positive fundamental solution of the heat equation

$$\partial_t u(x,t) = \Delta u(x,t) \quad (x,t) \in M \times (0,\infty),$$

where Δ is the Laplace-Beltrami operator on M. It is well-known that the heat kernel $p(t,x,y)$ can be also regarded as the transition density of the Brownian motion $(\{X_t\}_{r\geq 0}, \{\mathbb{P}_x\}_{x\in M})$ generated by Δ on M, namely, for any Borel set $A \subset M$,

$$\mathbb{P}_x(X_t \in A) = \int_A p(t,x,y)d\mu(y),$$

where μ is the Riemannian measure on M.

On $\mathbb{R}^n$, the heat kernel is given by the Gaussian function:

$$p(t,x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{\|x-y\|^2}{4t}}.$$

On a general manifold, the heat kernel is sensitive to the underlying space and the dependence has been an important subject in probability theory, harmonic analysis and geometry. In particular, on a non-compact Riemannian manifold, long time behavior of the heat kernel has been investigated under various settings by many authors. For the detail, see [2; 5; 15] and references therein.

Recently, Grigor'yan and Saloff-Coste [10], Grigor'yan, and Saloff-Coste and the author [8] proved heat kernel estimates on manifolds with ends. Let us quickly review these results. Let $M_1, \ldots, M_k$ be non-compact manifolds and for $1 \leq l \leq k$, K_l a compact set of M_l. A manifold M is called a connected sum of $M_1, \ldots, M_k$ with a central part $K \subset M$ if

$$M\backslash K = \bigsqcup_{l=1}^{k} E_l = \bigsqcup_{l=1}^{k} (M_l \backslash K_l).$$

Then the connected sum M is denoted by $M_1 \# M_2 \# \cdots \# M_k$. To study the behavior of the heat kernel on connected sums, parabolicity/non-parabolicity of the manifold plays an important role. Here, a manifold is *parabolic* if the Brownian motion is recurrent and *non-parabolic* if the Brownian motion is transient. Various equivalent conditions to parabolicity/non-parabolicity are known. See [3] for the detail.

When the central part K is compact and the all ends are non-parabolic together with a suitable condition (Poincaré inequality (PI), volume doubling property (VD) and relatively connected annuli condition (RCA)), behavior of the heat kernel is proved in [10]. For the detail of these conditions, see for instance [9; 11; 12; 14; 15] and references therein. Here we note only that Grigor'yan [6] and Saloff-Coste [14] proved that the condition (PI) together with (VD) is equivalent to the condition of the manifold that Li — Yau type heat kernel estimates (two-sided Gaussian estimates) holds:

$$p(t,x,y) \asymp \frac{C}{V(x,\sqrt{t})} e^{-b\frac{d^2(x,y)}{t}}, \tag{1}$$

where $V(x,r)$ is the Riemannian measure of a geodesic ball $B(x,r)$ and the symbol $\asymp$ means that both $\leq$ and $\geq$ hold by changing constants $C > 0$ and $b > 0$.

For $x \in E_i \subset M_i$, denote by $V_i(x,r)$ the Riemannian measure of a geodesic ball $B_i(x,r)$ in M_i. Let

$$V_{\min}(r) = \min_{1\leq l\leq k} V_l(r) = \min_{1\leq l\leq k} V_l(o_l, r),$$

where $o_l \in K_l$ is a reference point on each M_l in K_l. For $x \in E_i$ and $t > 0$, let $|x| = d(x, K) + 1$ and we define a function $H(x,t)$ as

$$H(x,t) = \min\left\{1, \frac{|x|^2}{V_i(|x|)} + \left(\int_{|x|^2}^{t} \frac{ds}{V_i(\sqrt{s})}\right)_+\right\}.$$

Then the following heat kernel estimates between different ends are obtained:

Theorem 1.1 (Grigor'yan and Saloff-Coste [10, Theorems 4.9 and 5.10]). *Let M be a connected sum of non-parabolic manifolds $M_1, M_2, \ldots, M_k$ with (PI), (VD) and (RCA). For $x \in E_i$, $y \in E_j$ with $i \neq j$ and $t > 0$,*

$$p(t,x,y) \asymp C\left(\frac{H(x,t)H(y,t)}{V_{\min}(\sqrt{t})} + \frac{H(y,t)}{V_i(\sqrt{t})} + \frac{H(x,t)}{V_j(\sqrt{t})}\right) e^{-b\frac{d^2(x,y)}{t}}. \tag{2}$$

In particular, for $x \in E_i$, $y \in E_j$ with $|x| \approx r$ and $|y| \approx 1$ (resp. $|x| \approx 1$ and $|y| \approx r$) (center-middle time regime), we obtain

$$p(r^2,x,y) \asymp \frac{C}{V_i(r)} \quad \left(\text{resp. } \frac{C}{V_j(r)}\right). \tag{3}$$

For $x \in E_i$ and $y \in E_j$ with $|x| \approx |y| \approx r$ (middle time regime), we have

$$p(r^2,x,y) \asymp C\frac{r^2}{V_i(r)V_j(r)} \ll \frac{C}{V_{\max}(r)}, \tag{4}$$

where $V_{\max}(r) = \max_{1\le l\le k} V_l(r)$. These estimates motivate us to observe the bottleneckness of the space in terms of the heat kernel behavior. We note that, in these regimens (center-middle time regime, middle time regime), the effect of other ends (first term in (2)) to the heat kernel is dominated.

When the connected sum is non-parabolic but there exist some parabolic ends (mixed case), Grigor'yan and Saloff-Coste proved the heat kernel estimates by using Doob's transform. For $1 \le l \le k$, set

$$h_l(r) = 1 + \left(\int_1^{r^2} \frac{ds}{V_l(\sqrt{s})}\right)_+.$$

For $x \in E_i$ and $r > 0$, set also

$$\widetilde{V}_i(x,r) = \left(h_i^2(|x|) + h_i^2(r)\right) V_i(x,r)$$

and for $t > 0$,

$$\widetilde{H}(x,t) := \frac{|x|^2}{h_i^2(|x|)V_i(|x|)} + \frac{1}{h_i(|x|)h_i(\sqrt{t})}\left(\int_{|x|^2}^{t} \frac{ds}{V_i(\sqrt{s})}\right)_+.$$

Then the following heat kernel estimates between different ends are obtained:

Theorem 1.2 (Grigor'yan and Saloff-Coste [10, Theorem 6.5 and Remark 6.7]). *Let M be a connected sum of manifolds with (PI), (VD) and (RCA) and at least one end is non-parabolic. Then, for $x \in E_i$ and $y \in E_j$ with $i \neq j$ and $t > 0$,*

$$p(t,x,y) \asymp Ch_i(|x|)h_j(|y|)\left(\frac{\widetilde{H}(x,t)\widetilde{H}(y,t)}{\widetilde{V}_{\min}(\sqrt{t})} + \frac{\widetilde{H}(x,t)}{\widetilde{V}_j(\sqrt{t})} + \frac{\widetilde{H}(y,t)}{\widetilde{V}_i(\sqrt{t})}\right) e^{-b\frac{d^2(x,y)}{t}}.$$

Let us observe heat kernel behavior in some typical regimes. For simplicity, let us consider the case that $V_l(r) \approx r^{\alpha_l}$ (see [10, Example 6.11 and Corollary 6.12]).

If $2 < \alpha_i, \alpha_j$, the estimate is the same as the case that all ends are non-parabolic stated in (3), (4).

If $0 < \alpha_i, \alpha_j < 2$, for $x \in E_i$ and $y \in E_j$ with $|x| \approx r$, $|y| \approx 1$ (resp. $|x| \approx 1$ and $|y| \approx$), we obtain

$$p(r^2, x, y) \asymp \frac{C}{r^{4-\alpha_i}} \ll \frac{C}{V_i(r)} \quad \left(\text{resp. } \frac{C}{r^{4-\alpha_j}} \ll \frac{C}{V_j(r)}\right). \tag{5}$$

For $x \in E_i$ and $y \in E_j$ with $|x| \approx |y| \approx r$, we obtain

$$p(r^2, x, y) \asymp \frac{C}{r^{6-\alpha_i-\alpha_j}} \ll C\frac{r^2}{V_i(r)V_j(r)}.$$

In this situation, the estimate in (5) (center-middle time regime) shows that the heat kernel behavior on each one end is far different from the Li — Yau type bound in (1).

If $\alpha_i < 2 < \alpha_j$, for $x \in E_i$ and $y \in E_j$ with $|x| \approx r$ and $|y| \approx 1$ (resp. $|x| \approx 1$ and $|y| \approx r$),

$$p(r^2, x, y) \asymp \frac{C}{r^{4-n_i}} \ll \frac{C}{V_i(r)} \quad \left(\text{resp. } \frac{C}{r^{n_j}} \approx \frac{C}{V_j(r)}\right). \tag{6}$$

For $x \in E_i$ and $y \in E_j$ with $|x| \approx |y| \approx r$, we obtain

$$p(r^2, x, y) \asymp \frac{C}{r^{2+\alpha_i-\alpha_j}} \ll C\frac{r^2}{V_i(r)V_j(r)}.$$

In this situation, similar to the previous case, the estimate in (6) with $|x| \approx r$ and $|y| \approx 1$ shows us the heat kernel behavior on E_i is different from the Li — Yau type bound in (1).

When the connected sum is parabolic (i.e., all ends are parabolic), the heat kernel estimates are obtained in [8]. To state the result, we prepare some notation. A parabolic manifold M is called *critical* if $V(x,r) \approx r^2$ and *subcritical* if

$$h(r) = \int_1^{r^2} \frac{ds}{V(\sqrt{s})} \le C\frac{r^2}{V(r)}.$$

We remark that $\mathbb{R}^2$ is critical and $\mathbb{R}$ is subcritical.

Then the following heat kernel estimates between different ends are obtained:

Theorem 1.3 (Grigor'yan, Ishiwata, Saloff-Coste [8, Theorem 2.3]). *Let M be a connected sum of parabolic manifolds with (PI), (VD) and (RCA). If all ends are subcritical, then for $x \in E_i$ and $y \in E_j$,*

$$p(t, x, y) \asymp \frac{C}{V_{\max}(\sqrt{t})} e^{-b\frac{d^2(x,y)}{t}}.$$

This shows that even if $x \in E_i$ and $y \in E_j$ are far enough from the center, the heat kernel is similar to the on-diagonal bound in the central part. Hence, in this situation, the bottleneckness of the space cannot be seen in the heat kernel estimates in the middle time regime.

When some ends are critical, we can see a bottleneck effect in the behavior of the heat kernel, but the effect is milder than that on non-parabolic connected sums. For example, consider $M = \mathbb{R}^2 \# \mathbb{R}^2$. Then, for $x \in E_1$ and $y \in E_2$ with $|x| \approx |y| \approx r$,

$$p(r^2, x, y) \asymp \frac{C}{r \log r},$$

which is slightly smaller than the on-diagonal bound $\frac{C}{r}$.

2. Main results

In view of the above observations of the heat kernel behavior on connected sums, we consider an inverse problem, which asks a bottleneck structure of a manifold appearing in the heat kernel estimates.

The condition of the manifold we need to observe the bottleneck structure is as follows. First, from the heat kernel estimates on connected sums of parabolic manifolds in Theorem 1.3, the manifold is required to be non-parabolic. Take a non-parabolic manifold divided into two domains by a partition. From the heat kernel estimates in the center-middle time regime (3), (5), our second requirement of the space is that the heat kernel on each one domain (including near the partition) is well-known (i.e. Li — Yau type bound (1)). Under these settings of the underlying space, our interest is a bottleneck structure of the space appearing in a heat kernel behavior between different domains.

The bottleneckness of the space is characterized by the capacity of closed sets and the first non-zero Neumann eigenvalue on balls. For a non-empty closed set F on M, the capacity of the capacitor (F, M) is defined by

$$\operatorname{cap}(F) := \inf_{\substack{f \in \operatorname{Lip}_0(M),\\ f = 1 on F}} \int |\nabla f|^2 d\mu,$$

where $\operatorname{Lip}_0(M)$ is spaces of all compactly supported Lipschitz functions on M. For an open set $\Omega \subset M$, let $\lambda_1^N(\Omega)$ be the first non-zero Neumann eigenvalue on Ω, namely, by definition, $\lambda_1^N(\Omega)$ is the minimum positive vaule so that there exist $f \in \mathcal{D} := C^2(\Omega) \cap C^1\left(\overline{\Omega}\right)$ such that

$$\begin{cases} \Delta f = \lambda f & \text{in } \Omega, \\ \nu f = \mathbf{0} & \text{on } \partial\Omega, \end{cases}$$

where ν is the normal derivative on $\partial\Omega$. By max-min theorem (c.f. [1, p. 17]),

$$\lambda_1^N(\Omega) = \inf_{f \in \mathcal{D}} \frac{\int_\Omega |\nabla f|^2 d\mu}{\int_\Omega |f - f_\Omega|^2 d\mu},$$

where $f_\Omega = \frac{1}{\mu(\Omega)} \int_\Omega f d\mu$. We note that the (strong) Poincaré inequality (PI) is rewritten by

$$\lambda_1^N(B(x, r)) \geq \frac{c}{r^2}.$$

For a subset $A \subset M$ and $\delta > 0$, let $A_\delta = B(A, \delta)$, that is, the δ-open neighborhood of A. We call $A_\delta \backslash A$ the δ-*skin of* A. Then we obtain the following.

Theorem 2.1. *Let M be a geodesically complete non-parabolic manifold divided into two unbounded connected components E_1 and E_2 by a closed set $F \subset M$. Fix a refernce point $o \in F$ and we assume that there exist curves $x_r \in E_1$ and $y_r \in E_2$ satisfying the following conditions:*

(i) There exist constants $a, \delta > 0$ such that for all $r > d(o, \partial F)$, $r^2/4 \le s \le r^2$ and for all $z \in F_\delta(o,r) := \overline{F_\delta \backslash F \cap B(o,r)}$,

$$p(s, x_r, z) \ge \frac{a}{V_1(x_r, r)},$$
$$p(s, y_r, z) \ge \frac{a}{V_2(y_r, r)},$$

where $V_l(x,r) = \mu(B(x,r) \cap E_l)$.

(ii) There exists a positive valued function $D(r)$ such that

$$p(r^2, x_r, y_r) \le \frac{D(r)}{V_1(x_r, r) V_2(y_r, r)}.$$

(See Fig. 1).

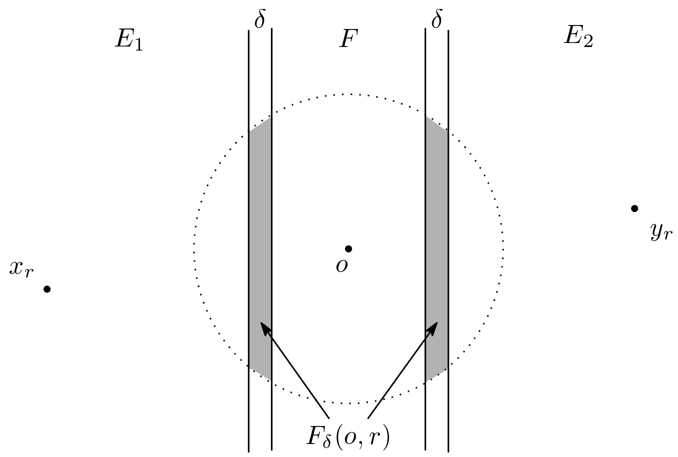

Fig. 1. Partition of M, x_r, y_r and $F_\delta(o,r)$

Then we obtain

$$\operatorname{cap}(F_\delta(o,r)) \le \frac{4D(r)}{a^2 r^2}. \tag{7}$$

This concludes also the upper bound of the first non-zero Neumann eigenvalue λ_1^N on balls. Fix $x \in M$ and $r > \max_{i=1,2} d(x, E_i)$. For all $0 < \varepsilon < 1$, set

$$E_1(\varepsilon) := \{z \in E_1 \ : \ \mathbb{P}_z(\tau_{F_\delta(x,r)} < \varepsilon)\}, \quad E_2(\varepsilon) := \{z \in E_2 \ : \ \mathbb{P}_z(\tau_{F_\delta(x,r)} < \varepsilon)\}. \tag{8}$$

Then we obtain

$$\lambda_1^N(B(x,r)) \le \frac{8D(d(x,o)+r)}{a^2(1-\epsilon)^2(d(x,o)+r)^2 \min_{i=1,2} \mu(B(x,r) \cap E_i(\epsilon))}. \tag{9}$$

Corollary 2.2. *Under the same setting of the above theorem, assume further that the isoperimetric inequality with isoperimetric function I holds:*

$$\mu_{n-1}(\partial\Omega) \geq I(\mu(\Omega)) \quad \forall \Omega \subset M,$$

where $\mu_{n-1}(\partial\Omega)$ *is the induced measure of the boundary of* Ω*. Then by [4; 13],* $\mu(F_\delta(o,r))$ *satisfies the following inequality:*

$$\left(\int_{\mu(F_\delta(o,r))}^{\infty} \frac{dv}{I^2(v)}\right)^{-1} \leq \mathrm{cap}(F_\delta(o,r)) \leq \frac{4D(r)}{c^2 r^2}.$$

In particular, if I satisfies

$$\int^{\infty} \frac{dv}{I^2(v)} < \infty \tag{10}$$

and $D(r) = Cr^2$*, we obtain*

$$\mu(F_\delta(o,r)) \leq C'$$

for some constant $C' > 0$ *and for all* $r > 0$*, namely,*

$$\mu(F_\delta \backslash F) < \infty.$$

As a more general situation, let us consider the case $D(r) = Cr^\alpha$ *with some* $\alpha \geq 2$ *and* $I(v) = v^{\frac{n-1}{n}}$*. Then we obtain*

$$\mu(F_\delta(o,r)) \leq C' r^{\frac{n}{n-2}(\alpha-2)}. \tag{11}$$

We note that the boundedness (10) implies the non-parabolicity of M (see [4, Section 3]). Moreover, we note also that the estimate in (11) may not be optimal. See Example 2.4 below.

Let us apply these results to some examples.

Example 2.3 (Non-parabolic connected sum with compact central part). Let M is a connected sum of manifolds $M_1, M_2, \ldots, M_k$ with (PI), (VD) and (RCA) along a compact central part K. Assume that there exits at least one non-parabolic end. As a partition, take $F = K \sqcup E_3 \cdots \sqcup E_k$ and choose curves $x_r \in E_1$ and $y_r \in E_2$ so that $d(x_r, K) \approx d(y_r, K) \approx r$.

If M_1 and M_2 are non-parabolic, then the condition (i) in Theorem 2.1 holds by (3) and (ii) is also true with $D(r) = Cr^2$ by (4). Then Theorem 2.1 asserts that

$$\mathrm{cap}(F_\delta(o,r)) \leq \frac{4C}{a^2}$$

for any $r > 0$, namely,

$$\mathrm{cap}(F_\delta \backslash F) \leq \frac{4C}{a^2}.$$

Moreover, for all $x \in M$ and for sufficiently large $r > \max_{i=1,2} d(x, E_i)$, we obtain by (9)

$$\lambda_1^N(B(x,r)) \leq \frac{C'}{V_{\min}(r)} \ll \frac{C'}{r^2}$$

for some constant $C' > 0$, which fails the Poincaré inequality (PI). We note that in this situation, we cannot observe other ends.

If one of M_1, M_2 is parabolic, then the condition (i) is failed by the estimates in (5) and (6).

Next, let us verify the effect of Theorem 2.1 on a connected sum with a non-compact central part.

Example 2.4 (Connected sum of two copies of $\mathbb{R}^n$ along a surface of revolution (c.f. [7])). For $n \geq 3$, let M be a connected sum of two copies of $\mathbb{R}^n$ along the surface of revolution $A(m, \alpha)$ $(0 \leq m \leq n-2,\ 0 \leq \alpha < 1)$ defined by

$$A(m, \alpha) = \{x \in \mathbb{R}^n \ : \ h(x) \leq r(x)^\alpha\},$$

where

$$r(x) = \left(1 + \sum_{1 \leq i \leq \leq m} x_i^2\right)^{1/2}, \quad h(x) = \left(\sum_{m+1 \leq i \leq n} x_i^2\right)^{1/2}.$$

In the case $m = 0$, we always take $\alpha = 0$.

As a partition F of the space, take the joint part along the surface of $A(m, \alpha)$. We choose continuous curves x_r in E_1 and y_r in E_2) given by

$$x_r = y_r = (\underbrace{0, \ldots, 0}_{m}, 1 + r, 0, \ldots, 0),$$

which is vertical to the subspace $\mathbb{R}^m \subset \mathbb{R}^n$ (see Fig. 2).

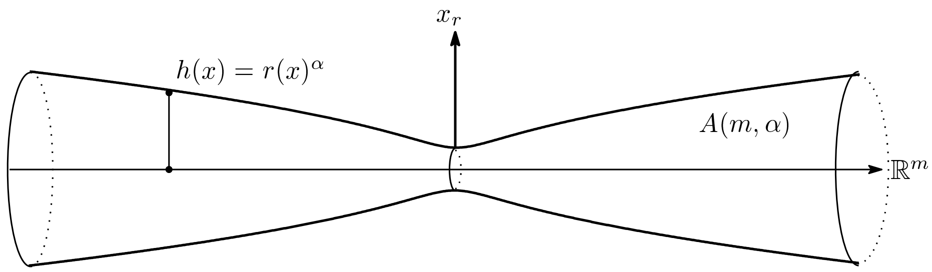

Fig. 2. $A(m, \alpha) \subset \mathbb{R}^n$ and x_r

Then, by using Theorem 1.1 in [7], for any $z \in F_\delta(o, r)$, we obtain

$$p(r^2, x_r, z) \asymp p(r^2, y_r, z) \asymp \frac{a}{r^n}$$

and

$$p(r^2, x_r, y_r) \asymp \frac{C}{r^{n+(1-\alpha)(n-m-2)}},$$

whence the conditions (i) and (ii) in Theorem 2.1 are true with $D(r) = Cr^{n-(1-\alpha)(n-m-2)}$. Then Theorem 2.1 asserts that

$$\mathrm{cap}(F_\delta(o, r)) \leq \frac{C}{a^2} r^{m+\alpha(n-m-2)},$$

which gives an optimal estimate of the capacity growth of the δ-skin of the surface of $A(m, \alpha)$. Moreover, for all $x \in M$ and sufficiently large $r > \max_{i=1,2} d(x, E_i)$, we obtain by (9) that

$$\lambda_1^N(B(x, r)) \leq \frac{C'}{r^{2+(1-\alpha)(n-m-2)}} \ll \frac{C'}{r^2}$$

for some constant $C' > 0$, which fails the Poincaré inequality (PI).

Since the isoperimetric inequality with $I(v) = cv^{\frac{n-1}{n}}$ is true (see [7, Section 3]), Corollary 2.2 asserts that

$$\mu(F_\delta(o,r)) \le C' r^{\alpha n + \frac{nm}{n-2}(1-\alpha)}$$

while

$$\mu(F_\delta(o,r)) \approx r^{m+\alpha(n-m-1)} << C' r^{\alpha n + \frac{nm}{n-2}(1-\alpha)}.$$

3. Proof of Theorem 2.1

The following lemma is a key to prove Theorem 2.1.

Lemma 3.1 (c.f. [10; 11]). *Let M be a non-parabolic manifold. For any compact set $F \subset M$ and for all $x, y \notin F$ and for all $t > 0$,*

$$p(t,x,y) \ge \frac{t}{4}\mathrm{cap}(F) \inf_{\substack{z\in\partial F\\ t/4\le s\le t}} p(s,x,z) \inf_{\substack{z\in\partial F\\ t/4\le s\le t}} p(s,z,y).$$

Proof. The proof is similar to Lemma 3.1 in [10]. Let τ_F be the first hitting time to F of a Brownian path ω, namely

$$\tau_F(\omega) = \inf\{t \ge 0 \,:\, X_t(\omega) \in F\}.$$

We denote by $\psi_F(t,x)$ the hitting probability

$$\mathbb{P}_x(\tau_F < t).$$

Let $\mathbb{E}_x$ be the expectation of the Brownian motion on M. For any Borel set $A \subset M$, we have

$$\begin{aligned}
\mathbb{P}_x(X_t \in A) = \int_A p(t,x,y)d\mu(y) = \mathbb{E}_x(\delta_A(X_t)) \ = \\
= \mathbb{E}_x\left(\left(1_{\{\tau_F\le t/2\}} + 1_{\{t/2<\tau_F\le t\}} + 1_{\{\tau_F>t\}}\right)\delta_A(X_t)\right) \ = \\
= \mathbb{E}_x\left(1_{\{\tau_F\le t/2\}}\delta_A(X_t)\right) + \mathbb{E}_x\left(1_{\{t/2<\tau_F\le t\}}\delta_A(X_t)\right) \ + \\
+ \mathbb{E}_x\left(1_{\{\tau_F>t\}}\delta_A(X_t)\right) \ \ge \\
\ge \mathbb{E}_x\left(1_{\{\tau_F\le t/2\}}\delta_A(X_t)\right),
\end{aligned}$$

where δ_A is the characteristic function of A. By the strong Markov property of the Brownian motion on M, we obtain

$$\begin{aligned}
\mathbb{E}_x\left(1_{\{\tau_F\le t/2\}}\delta_A(X_t)\right) = \mathbb{E}_x\left(1_{\{\tau_F\le t/2\}}\mathbb{E}_{X_{\tau_F}}\left(\delta_A(X_{t-\tau_F})\right)\right) \ = \\
= \mathbb{E}_x\left(1_{\{\tau_F\le t/2\}}\mathbb{P}_{X_{\tau_F}}(X_{t-\tau_F} \in A)\right) \ \ge \\
\ge \psi_F\left(\frac{t}{2}, x\right) \inf_{\substack{t/2\le s\le t\\ z\in\partial F}} \mathbb{P}_z(X_s \in A) \ \ge \\
\ge \int_A \psi_F\left(\frac{t}{2}, x\right) \inf_{\substack{t/2\le s\le t\\ z\in\partial F}} p(s,z,y)d\mu(y).
\end{aligned}$$

Since $A \subset M$ is arbitrary, this concludes that

$$p(t,x,y) \geq \psi_F\left(\frac{t}{2},x\right) \inf_{\substack{t/2\leq s\leq t\\ z\in\partial F}} p(s,z,y).$$

Since M is non-parabolic and F is compact, we obtain by Theorem 3.7 in [11] that for any $x \notin F$,

$$\psi_F\left(\frac{t}{2},x\right) \geq \mathrm{cap}(F)\int_0^{t/2} \inf_{z\in\partial F} p(s,z,x)ds \geq \frac{t}{4}\mathrm{cap}(F) \inf_{\substack{t/4\leq s\leq t/2\\ z\in\partial F}} p(s,z,x),$$

which concludes the lemma.

Proof of Theorem 2.1. Substituting the assumptions (i) and (ii) of the heat kernel into the above lemma for $F = F_\delta(o,r)$ and $t = r^2$, we obtain

$$\frac{D(r)}{V_1(x_r,r),V_2(y_r,r)} \geq \frac{r^2}{4}\mathrm{cap}(F_\delta(o,r))\frac{a}{V_1(x_r,r)}\frac{a}{V_2(y_r,r)}$$

which concludes (7).

Next we prove the upper bound of the first non-zero Neumann eigenvalue (9). For $x \in M$ and $r > \max_{i=1,2} d(x,E_i)$, choose a test function $f \in C^\infty(B(x,r))$ as

$$f(x) = \begin{cases} 1-\Psi_{F_\delta(x,r)}(x) & \text{if } x\in B(x,r)\cap E_1\\ 0 & \text{if } x\in B(x,r)\cap K\\ -c\left(1-\Psi_{F_\delta(x,r)}(x)\right) & \text{if } x\in B(x,r)\cap E_2, \end{cases} \tag{12}$$

where $\Psi_{F_\delta(x,r)}(x) = \Psi_{F_\delta(x,r)}(\infty,x)$ and $c = c(r)$ is a positive constant so that $f_{B(x,r)} = 0$. Since the hitting probability $\Psi_F(x)$ is the equilibrium potential of the capacitor (F,M), that is,

$$\mathrm{cap}(F) = \int_M |\nabla\Psi_F|^2 d\mu,$$

we obtain

$$\begin{aligned}\lambda_1^N(B(x,r)) \leq& \frac{\int_{B(x,r)}|\nabla f|^2 d\mu}{\int_{B(x,r)}|f|^2 d\mu}\\ \leq& \frac{\mathrm{cap}\left(F_\delta(x,r)\right) + c^2\mathrm{cap}\left(F_\delta(x,r)\right)}{\int_{B(x,r)\cap E_1}|1-\Psi_{F_\delta(x,r)}|^2 d\mu + c^2\int_{B(x,r)\cap E_2}|1-\Psi_{F_\delta(x,r)}|^2 d\mu}.\end{aligned}$$

By the definitions of the test function f in (12) and $E_i(\varepsilon)$ in (8), for $i = 1,2$,

$$\int_{B(x,r)\cap E_i} |1-\Psi_{F_\delta(x,r)}|^2 d\mu \geq \int_{B(x,r)\cap E_i(\varepsilon)} |1-\Psi_{F_\delta(x,r)}|^2 d\mu \geq (1-\epsilon)^2\mu(B(x,r)\cap E_i(\varepsilon)).$$

Then we obtain

$$\lambda_1^N(B(x,r)) \leq \frac{(1+c^2)\mathrm{cap}(F_\delta(x,r))}{(1-\epsilon)^2\left[\mu(B(x,r)\cap E_1(\varepsilon)) + c^2\mu(B(x,r)\cap E_2(\varepsilon))\right]}. \tag{13}$$

If $c \geq 1$, the estimate (13) implies that

$$\lambda_1^N(B(x,r)) \leq \frac{2c^2 \mathrm{cap}(F_\delta(x,r))}{(1-\epsilon)^2 c^2 \mu(B(x,r) \cap E_2(\varepsilon))} = \frac{2\mathrm{cap}(F_\delta(x,r))}{(1-\epsilon)^2 \mu(B(x,r) \cap E_2(\varepsilon))}.$$

If $0 < c < 1$, we obtain

$$\lambda_1^N(B(x,r)) \leq \frac{2\mathrm{cap}(F_\delta(x,r))}{(1-\epsilon)^2 \mu(B(x,r) \cap E_1(\varepsilon))}.$$

Since

$$F_\delta(x,r) \subset F_\delta\left(o, d(x,o) + r\right),$$

the estimate in (7) concludes that

$$\lambda_1^N(B(x,r)) \leq \frac{8D\left(d(x,o)+r\right)}{a^2(1-\epsilon)^2(d(x,o)+r)^2 \min_{i=1,2} \mu(B(x,r) \cap E_i(\epsilon))},$$

whence the proof of Theorem 2.1 is completed.

REMARK

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МОЖНО ЛИ ПО РАСПРЕДЕЛЕНИЮ ТЕПЛА СУДИТЬ О НАЛИЧИИ «БУТЫЛОЧНОГО ГОРЛЫШКА» У ПРОСТРАНСТВА?

Сатоши Ишивата

Профессор, Факультет математических наук,
Yamagata University
ishiwata@sci.kj.yamagata-u.ac.jp
Yamagata 990-8560, Japan

Аннотация. В этой статье мы обсуждаем наличие узких мест в структуре некомпактного многообразия, проявляющееся в поведении ядра уравнения теплопроводности. Родственная обратная задача оценки ядра уравнения теплопроводности на многообразиях с концами изучалась в [8; 10]. В результате, если непараболическое многообразие делится на две области и имеются подходящие оценки ядра уравнения теплопроводности между разными областями, то мы получаем верхнюю оценку роста емкости δ-*skin* разбиения. По этой оценке емкости получаем верхнюю оценку первого ненулевого собственного числа Неймана оператора Лапласа — Бельтрами на шарах. В предположении изопериметрического неравенства также получен верхний предел роста объема δ-skin разбиения.

Ключевые слова: ядро уравнения теплопроводности, многообразие с концами, обратная задача.