www.volsu.ru

МАТЕМАТИКА И МЕХАНИКА

DOI: https://doi.org/10.15688/mpcm.jvolsu.2019.1.1

UDC 517.442
LBC 511

THE REGULARITY OF THE LAPLACE TRANSFORM

Andrey Valerianovich Pavlov
Candidate of Physical and Mathematical Sciences, Associate Professor,
Department of Higher Mathematics-1,
MIREA — Russian Technological University
a_pavlov@mirea.ru
Prosp. Vernadskogo, 78, 119454 Moscow, Russian Federation

Abstract. The paper proves the regularity of the double Laplace transform in the neighborhood of zero. The class of the transform of Laplace from the transform of Fourier is considered from the functions without a regularity in null.

Key words: Transform of Fourier, transform of Laplace, regularity of the double transform of Laplace, regularity of the transform of Laplace from the transform Fourier.

In the memory of B.V. Gnedenko and A.D. Solovjev

1. Introduction

We consider the regularity of the double transform of Laplace (the theorem 1, the remarks 1, 2). The theorem proved in this part have a general-mathematical character and are easily checked up. With help of the theorem 1 and the remark 2 it is simply to prove a some theorems related with the transform of Fourier and Laplace [3–6] (for instance, about the inverse operator of the transform of Laplace, using only positive values of the transform of Laplace on the $[0, +\infty)$ [5]). The theorems are not by the theme of the the article and require the separate study in connection with the theorem 1. Some results in the direction were formulated in the works [4–7].

The fact about double decomposition on the elementary fractions is considered conclusion. In opinion of author the fact underlines interest to the remark 2.

By definition,

$$L_{\pm}Z(t)(x) = \int\limits_0^{\infty} e^{\pm xt} Z(t)dt, x \in [0, \infty),$$

(we will use $L_{\pm}Z(t)(\cdot)(x) = L_{\pm}Z(t)(x)$ too),

$$F_{\pm}u(t)(\cdot)(p) = \int\limits_{-\infty}^{\infty} e^{\pm pit} u(t)dt, p \in (-\infty, \infty), L_{+} = L,$$

$$C^0 u(t)(\cdot)(x) = \int\limits_0^{\infty} \cos xtu(t)dt, S^0 u(t)(\cdot)(x) = \int\limits_0^{\infty} \sin xtu(t)dt, x \in (-\infty, \infty),$$

$$F^0_{\pm}u(t)(\cdot)(p) = \int\limits_0^{\infty} e^{\pm pit} u(t)dt, p \in (-\infty, \infty).$$

2. The regularity of Laplace transform in $|z| < a > 0$

In the section we use the Y1 condition.

The **Y1 condition** takes place for the $u(p)$ function, if the $u(p)$ function is regular for all p without only k points $z_1, \ldots, z_k, z_j \notin (-\infty, \infty) \bigcup (-i\infty, i\infty), k = 0, 1, \ldots,$ $u(0) = 0$, and

$$\max[|u(p)|, |du(p)/dp|, |d^2 u(p)/p^2|]|p^{2+\delta}| \to 0, |p| \to \infty,$$

$\delta > 0$, $\delta = \text{const}$.

We use the Ch1 condition too.

Ch1 condition.

The $u(p)$ function is regular in $K_{++} = \{p : Im\, p \geq 0\} \bigcap \{p : Re\, p \geq 0\}$ or in $K_{+-} = \{p : Im\, p \leq 0\} \bigcap \{p : Re\, p \geq 0\}$.

Theorem 1. *The functions*

$$LF^0_{+}u(x)(\cdot)(z) = \int\limits_0^{\infty} e^{-zt}dt \int\limits_0^{\infty} e^{itx}u(x)dx = iLLu(x)(\cdot)(iz),\ \ LLu(x)(\cdot)(z),$$

are regular in the area $\{z : |z| < \varepsilon\}$ for some $\varepsilon > 0$, if for the $u(p)$ function the Y1 condition takes place, and $Re\, u(t) = u(t), t \in [0, \infty)$.

Proof. We can use the proposition 1 [5; 6].

Proposition 1. The equalities

$$LF^0_{+}u(x)(\cdot)(v) = iF^0_{-}Lu(x)(\cdot)(v), v \in [0, \infty),$$

$$LC^0 u(x)(\cdot)(v) = S^0 Lu(x)(\cdot)(v), LS^0 u(x)(\cdot)(v) = C^0 Lu(x)(\cdot)(v), v \in [0, \infty)$$

take place, if for the function $u(p)$ the Y1 condition takes place. The similar equality $LF^0_{-}u(x)(\cdot)(v) = -iF^0_{+}Lu(x)(\cdot)(v), v \in [0, \infty)$ takes place too.

Proof. We get the first formula after the change of order of integration in both parts of the first equality. If $u(0)=0$, it is obviously with help of the expressions

$$|F_-^0 u(x)(\cdot)(t)| \le |(du(x)/dx|_{x=0})/t^2| + |(1/t^2)F_-^0(d^2u(x)/dx^2)(\cdot)(t)| \le c_1/t^2, t \to \infty,$$

$c_1 = \text{const}$, $c_1 < \infty$ (see [8]).

With help of the proposition 1 we obtain, that the $F_-^0 Lu(x)(\cdot)(p) = l_-(p)$ function is defined for all $\operatorname{Im} p < 0$, and

$$\lim_{p \to iy, \operatorname{Im} p<0} F_-^0 Lu(x)(\cdot)(p) = F_-^0 Lu(x)(\cdot)(iy), y \in (-\infty, \infty),$$

(it is obviously, if $u(0)=0$; as in the proposition 1 we use the formula of integration on parts [8]).

Similar facts take place for a similar function $l_+(p) = F_+^0 Lu(x)(\cdot)(p)$; the function is definite from other side of plane $\operatorname{Im} p > 0$.

We suppose $u(-p) = -u(p)$.

We can write

$$F_-^0 Lu(x)(\cdot)(p) + F_+^0 Lu(x)(\cdot)(p) = 2C^0 Lu(x)(\cdot)(p) = F(p), \quad p = y, \quad y \in [0, \infty),$$

if $u(-p) = -u(p)$, or

$$l_-(p) + l_+(p) = F(p),$$

where $F(p)$ are regular in $\{p : |\operatorname{Im} p| < A\} \bigcup \{p : |\operatorname{Re} p| < A\}$ for some $A > 0$, if function $u(p)$ is regular as in the Y1 condition (the fact is well-known [2; 5; 6]).

To prove the fact for $p = y \in (-\infty, 0]$ we can define a new functions $l_-^{to+}(p), l_+^{to-}(p)$:

$$l_-^{to+}(p) = l_-(p), \operatorname{Im} p \le 0,$$

$$l_+^{to-} = l_+(p), \operatorname{Im} p \ge 0,$$

where $l_-^{to+}(p)$ is an analytical continuation of the $l_-(p), \operatorname{Im} p \le 0$ function from the lower part of plane to the overhead part of plane $\{p : Im\, p \ge 0\}$; $l_+^{to-}(p)$ is an analytical continuation of the $l_+(p), \operatorname{Im} p \ge 0$ function from from the overhead part of plane to the lower part of plane $\{p : Im\, p \le 0\}$ [2].

The equality $l_-^{to+}(p) + l_+(p) = F(p), Im\, p \ge 0$ repeats the main equality $l_-(p) + l_+(p) = F(p)$, but in the $\{p : \operatorname{Im} p \ge 0\}$ area; the equality $l_-(p) + l_+^{to-}(p) = F(p)$ repeats the main equality $l_-(p) + l_+(p) = F(p)$, but in the $\{p : Im\, p \le 0\}$ area, where the $F(p)$ function is regular in $\{p : |\operatorname{Im} p| < A\} \bigcup \{p : |\operatorname{Re} p| < A\}$ for the $u(-p) = -u(p)$ function [2]. We obtain, that

$$l_-^{to+}(p) + l_+(p) = l_-(p) + l_+^{to-}(p), p = y \in [0, \infty).$$

But the same equality takes place and for the $p = y \in (-\infty, 0]$ (we use, that both functions $l_-^{to+}(p) + l_+(p) = F(p), l_-(p) + l_+^{to-}(p) = F(p)$ are equal to the regular $F(p)$ function in different parts of the plane for the $u(-p) = -u(p)$ function [2]).

We get

$$l_-^{to+}(p) + l_+(p) = l_-(p) + l_+^{to-}(p), p = y \in (-\infty, \infty).$$

The functions $l_-^{to+}(p), l_+(p), l_-(p), l_+^{to-}(p)$ are the transforms of Laplace in area of definition, and the functions are regular in area of definition [2] (from the proposition 1) with values on the boundary. The $l_-^{to+}(p), l_+^{to-}(p))$ functions are regular in the area of regularity of the sums from the the lemma 1.

Lemma 1. *1. The function* $LF_+^0 u(x)(\cdot)(p) = iF_-^0 Lu(x)(\cdot)(p) = il_-(p) = il_-^{to+}(p)$ *is regular for all* $\{p : Im\, p \geq 0\}$ *(we consider the branch of the function, passing through* $\{p : p = iy, y = Im\, p > 0\}$ *and* $\{p : p = x, x = Re\, p > 0\}$*, where the* $F_-^0 Lu(x)(\cdot)(p))$ *values not defined), if for the* $u(p)$ *function or for the* $V_i(p)$ *functions the Y1,Ch1 conditions take place* $i = 1, 2$*,* $u(p) = V_1(p) + V_2(p)$ *(see the part 2 of the remark 2 too).*

2. The function $LF_-^0 u(x)(\cdot)(p) = -iF_+^0 Lu(x)(\cdot)(p) = -il_+(p) = l_+^{to-}(p)$ *is regular for all* $\{p : Im\, p \leq 0\}$ *(we consider the branch of the function, passing through* $\{p : p = iy, y = Im\, p < 0\}$ *and* $\{p : p = x, x = Re\, p > 0\}$*, where the* $F_+^0 Lu(x)(\cdot)(p))$ *values not defined), if for the* $u(p)$ *function or for the* $V_i(p)$ *functions the Y1, Ch1 conditions take place* $i = 1, 2$*,* $u(p) = V_1(p) + V_2(p)$ *(see the part 2 of the remark 2 too).*

Proof. If the $u(p)$ function is regular in $K_{++} = \{p : \text{Im}\, p \geq 0\} \bigcap \{p : \text{Re}\, p \geq 0\}$, after integration along the line $L_+ = L_1 \bigcup L_2 \bigcup L_3$ of the $(1/ip + z)u(z)$ function anticlockwise , $L_1 = [0, R]$, $L_2 = \{p : p = Re^{i\varphi}, 0 \leq \varphi \leq \pi/2\}$, $L_3 = [iR, 0]$, we obtain, that

$$l_-^{to+}(p) = l_-(p) = F_-^0 Lu(x)(\cdot)(p) = \int_0^\infty (1/ip + x)u(x)dx =$$

$$= \int_0^\infty (1/ip + ix_1)u(ix_1)dix_1 = (1/i)LLu(ix_1)(\cdot)(p), p \in (0, +\infty),$$

as for $\{p : Re\, p \notin (-\infty, 0)\}$ so as for all $\{p : Im\, p \geq 0\}$. We use the Y1, Ch1 conditions (the $u(p)$ function is regular in $K_{++} = \{p : Im\, p \geq 0\} \bigcap \{p : Re\, p \geq 0\}$; for $Im\, p = 0$ we use the $u(0) = 0$ condition for proof of continuity on the $(-\infty, \infty)$ axis with help of the proposition 1 [8]).

We obtain, that the such sum is regular for all $\{p : \text{Im}\, p \geq 0, \text{Re}\, ip \leq 0\}$, and the function

$$\int_0^\infty (1/ip + ix_1)u(ix_1)dix_1 = (1/i)LLu(x)(\cdot)(p), Re\, p \notin (-\infty, 0)$$

is regular in $\{p : \text{Im}\, p \geq 0, \text{Re}\, ip \leq 0\}$ with the values on the boundary line $\{p : \text{Im}\, p = 0, \text{Re}\, p \leq 0\}$ (with help of the formula of integration on parts as in proposition 1 [2; 8]).

For the function $F_+^0 Lu(x)(\cdot)(p) = l_+(p) = l_+^{to-}(p), \text{Im}\, p < 0$ we use $l_+^{to-}(p) = \overline{l_-^{to+}(\overline{p})}$, as the branches of the functions $l_+(p) = \overline{l_-(\overline{p})}$ [2] (with help of the formula $F_+^0 Lu(x)(\cdot)(p) = \overline{F_-^0 Lu(x)(\cdot)(\overline{p})}$ on the $[0, +\infty)$ line) by the theorem of Riemann about the analytical continuation across the $(-\infty, \infty)$ line [2], and the function $l_+(p) = l_+^{to-}(p) = \overline{l_-^{to+}(p)(\overline{p})}$ is defined and regular in $\{p : Im\, p \leq 0\}$ (or for the $u(p) = V_1(p) + V_2(p)$ function in connection with the part 2 of the remark 2 too) .

If the function $u(p)$ is regular in $K_{+-} = \{p : \text{Im}\, p \leq 0\} \bigcap \{p : \text{Re}\, p \geq 0\}$, after integration along the line $L_+ = L_1 \bigcup L_{2*} \bigcup L_{3*}$ of the $(1/ip + z)u(z)$ function anticlockwise, $L_1 = [0, R]$, $L_{2*} = \{p : p = \text{R}e^{i\varphi}, -\pi/2 \leq \varphi \leq 0\}$, $L_{3*} = [-iR, 0]$, we obtain, that

$$F_+^0 Lu(x)(\cdot)(p) = l_+(p) = -\int_0^\infty (1/ip - x)u(x)dx =$$

$$= -\int_0^{-\infty} (1/ip - ix_1)u(ix_1)dix_1 =$$

$$= \int_0^{+\infty} (1/ip + ix_2)u(-ix_2)dix_2 = LLu(-ix_1)(\cdot)(p), p \in (0, +\infty).$$

The further proof of lemma repeats proof of the first part (we use, that $\mathrm{Re}\, u(t) = = u(t), t \in [0, \infty)$, but $u(p) = V_1(p) + V_2(p)$).

With help of lemma 1 we get

$$|l_-^{to+}(p) + l_+(p)| \le C = \text{const}, \ \mathrm{Im}\ p \ge 0; \ \ l_-(p) + l_+^{to-}(p) \le C = \text{const}, \ \mathrm{Im}\ p \le 0,$$

$C < \infty$. Both sums $l_-^{to+}(p) + l_+(p)$, $l_-(p) + l_+^{to-}(p)$ are regular in area of definition and continuous on the boundary [2; 8].

We proved

$$l_-^{to+}(p) + l_+(p) = C, \quad C = \text{const}, \quad C < \infty$$

for all p (see [2]) and $2C^0Lu(x)(\cdot)(p) = F(p) = l_-^{to+}(p) + l_+(p) \equiv C_1 = \text{const}$, $C_1 < \infty$ [2; 6] or $l_-^{to+}(p) = -l_+(p)$ for all p including $p \in (-\infty, \infty)$.

We obtain, that the $F(p)$ function is regular [2] with the values $2C^0Lu(x)(\cdot)(p) = = F(p), p \in (-\infty, \infty)$.

We can use, that the function $LF_+^0u(x)(\cdot)(p) = iF_-^0Lu(x)(\cdot)(p)$ is regular in

$$\{p : |\,\mathrm{Re}\ p| < \varepsilon\} \bigcup \{p : |\,\mathrm{Im}\ p| < \varepsilon\}$$

for some $\varepsilon > 0$. It is well-known fact [2; 5; 6] in Y1 condition for the function $u(p)$.

We proved, that

$$F_-^0Lu(x)(\cdot)(p) = F(p) - F_+^0Lu(x)(\cdot)(p)$$

is the analytical continuation from from one side of plane on other [2].

For the $u(-p) = u(p)$ we can use

$$F_-^0Lu(x)(\cdot)(p) - F_+^0Lu(x)(\cdot)(p) = 2iS^0Lu(x)(\cdot)(p) = F(p), p = y, y \in [0, \infty),$$

further by analogy with the first part (with help of the lemma 1 and the $u(0) = 0$ condition).

The theorem 1 is proved.

From the theorem 1 we obtain the remark 1.

Remark 1. *The theorem 1 takes place for the functions $u(p) = v(p) + v(-p)$, $u(p) = v(p) - v(-p)$, if all the essential points [2] of the $v(p)$ function are placed or in K_{++} (or all the essential points are in K_{-+}).*

From the remark 1 we get the first part of the remark 2.

The second part is easily proved by the methods of the work [1; 6].

Remark 2.

1) *The theorem 1 takes place for the function $u(p) = (v_1(p) - v_1(-p)) + (v_2(p) - v_2(-p))$, if the essential points [2] of the function $v_i(p)$ are placed or in K_{++} or in K_{-+}, $i = 1, 2$, and for the $u(p)$ functions the conditions of the theorem 1 take place.*

2) *The function $u(-p) = -u(p)$ can be presented in the form $u(p) = (v_1(p) - v_1(-p)) + (v_2(p) - v_2(-p))$, if for the function $u(p)$ the Y1 condition takes place.*

We will get a similar result, if to apply the $z_1 = 1/z$ inversion and the $w = e^{i\varphi} z_1, \varphi = \pi/4$, function; we use, that the same result we obtain for the functions in the reverse order with $\varphi = -\pi/2$ (by analogy for $-\varphi$).

3. Conclusion

We will mark the fact about double decomposition on the elementary fractions:

$$p[\frac{1}{p-1} - \frac{1}{p+1}] = \frac{1}{p-1} + \frac{1}{p+1},$$

$$\frac{p}{(p-1)^2} - \frac{p}{p^2-1} = \frac{1}{(p-1)^2} + \frac{1}{p^2-1},$$

$p \neq 1, -1$. The fact in opinion of author underlines interest to the theorem 1 and to the consequences of the theorem 1.

Probably, the equality $|C^0 S^0 u(t)(\cdot)(x)| = |S^0 C^0 u(t)(\cdot)(x)|$ ensues from theorem 1 and remark 2, $x \in (0, \infty)$.

REFERENCES

1. Kolmogorov A.N., Fomin S.V. *Elements of the Theory of Functions and Functional Analysis*. Moscow, Nauka Publ., 1976. 544 p.

2. Lavrentiev M.A., Shabat B.V. *The Methods of Theory Functions of Complex Variable*. Moscow, Nauka Publ., 1987. 688 p.

3. Pavlov A.V. The Fourier Transform and New Inversion Formula of the Laplace Transform. *Math. notes*, 2011, vol. 90, no. 6, pp. 793-796.

4. Pavlov A.V. Reliable Prognosis of the Functions in the Form of Transformations of Fourier or Laplace. *Herald of MIREA*, 2014, vol. 3, no. 2, pp. 78-85.

5. Pavlov A.V. The New Inversion of Laplace Transform. *Journal of Mathem. and Syst. Scien.*, 2014, vol. 4, no. 3, pp. 197-201.

6. Pavlov A.V. About the Equality of the Transform of Laplace to the Transform of Fourier. *Issues of Analysis*, 2016, vol. 23, no. 1, pp. 21-30.

7. Pavlov A.V. Identical Service and the Odd or Even Transform of Laplace. *Conferen. Anal. and comp.meth. in probab. theor. and its appl., (ACMPT2017)*. Moscow, Moscow University named after M.V. Lomonosov, 2017, pp. 31-35.

8. Fihtengoltz G.M. *The Course of Differential and Integral Calculus*, vol. II. Moscow, Nauka Publ., 1969. 800 p.

РЕГУЛЯРНОСТЬ ПРЕОБРАЗОВАНИЯ ЛАПЛАСА

Андрей Валерианович Павлов

Кандидат физико-математических наук, доцент кафедры высшей математики-1,
МИРЭА — Российский технологический университет
a_pavlov@mirea.ru
просп. Вернадского, 78, 119454 г. Москва, Российская Федерация

Аннотация. Доказана регулярность двойного преобразования Лапласа в окрестности нуля. Рассматривается класс функций с нарушением регулярности в нуле, преобразование Лапласа от преобразования Фурье от которых регулярно в окрестности нуля.

Ключевые слова: преобразование Фурье, преобразование Лапласа, регулярность двойного преобразования Лапласа, регулярность преобразования Лапласа от преобразования Фурье.