DOI: https://doi.org/10.15688/mpcm.jvolsu.2019.4.3

UDC 514.7
LBC 22.151

Submitted: 04.06.2019
Accepted: 03.10.2019

ON PSEUDO-SLANT SUBMANIFOLDS OF NEARLY QUASI-SASAKIAN MANIFOLDS

Shamsur Rahman

PhD, Assistant Professor, Department of Mathematics,
Maulana Azad National Urdu University
shamsur@rediffmail.com
Polytechnic Satellite Campus Darbhanga, 846002 Bihar, India

Mohd Sadiq Khan

PhD, Professor, Department of Mathematics,
Shibli National College
msadiqkhan.snc@gmail.com
Uttar Pradesh, 276001 Azamgarh, India

Aboo Horaira

Researcher, Department of Mathematics,
Shibli National College
aboohoraria@gmail.com
Uttar Pradesh, 276001 Azamgarh, India

Abstract. The geometry of pseudo-slant submanifolds of nearly quasi Sasakian manifold is studied. It is proved that totally umbilical proper-slant submanifold of nearly quasi Sasakian manifold admits totally geodesic if the mean curvature vector $H \in \mu$. The integrability conditions of the distributions of pseudo-slant submanifolds of nearly quasi Sasakian manifold are also obtained.

Key words: nearly quasi-Sasakian manifold, slant submanifold, proper slant submanifold, pseudo-slant submanifold.

Introduction

The notion of a slant submanifolds as natural generallization of both holomorphic and totally real immersions was given by B.Y. Chen [6]. Latter many research articles have been appeared on the existence of these submanifolds in various known spaces. The properties of slant submanifolds of an almost contact metric manifolds were studied by A. Lotta [10].

L. Cabrerizo et. al [8] defined slant submanifolds of Sasakian manifolds. N. Papagiuc [12] introduced and studied the notion of semi-slant submanifolds of an almost Hermitian manifold. A. Carrizo [5; 7] defined hemi-slant submanifolds. The contact version of pseudo-slant submanifold in a Sasakian manifold have been studied by V.A. Khan et. al [11]. In [13] the authors studied nearly quasi-Sasakian manifold.

The purpose of the paper is to study the notion of pseudo-slant submanifold of nearly quasi-Sasakian manifold. In section 1 we recall some results and formula later use. In section 2 we define pseudo-slant submanifold of nearly quasi Sasakian manifold. In section 3 it is concern with the integrability of the distribution on the pseudo slant submanifolds of nearly quasi Sasakian manifold and obtain some characterizations. In section 4 we obtain a classification theorem for totally umbilical pseudo-slant submanifold M of nearly quasi Sasakian manifold $\bar{M}$.

1. Preliminaries

Let $\bar{M}$ be a real $(2n+1)$ dimensional differentiable manifold endowed with an almost contact metric structure (f, ξ, η, g), where f is a tensor field of type $(1,1)$, ξ vector field, η is a 1-form and g is Riemannian metric on $\bar{M}$ such that

$$\begin{gathered}(a)\quad f^2 = -I + \eta \otimes \xi, \quad (b)\quad \eta(\xi) = 1, \quad (c)\quad \eta \circ f = 0, \\ (d)\quad f(\xi) = 0, \quad (e)\quad \eta(X) = g(X, \xi), \\ (f)\quad g(fX, fY) = g(X, Y) - \eta(X)\eta(Y)\end{gathered} \tag{1}$$

for any vector field X, Y tangent to $\bar{M}$, where I is the identity on the tangent bundle $\Gamma\bar{M}$ of $\bar{M}$. An almost contact metric structure (f, ξ, η, g) on $\bar{M}$ is called quasi-Sasakian manifold if

$$(\bar{\nabla}_X f)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2}$$

where A a symmetric linear transformation field, $\bar{\nabla}$ denotes the Riemannian connection of g on $\bar{M}$. If in a addition to above relations

$$(\bar{\nabla}_X f)Y + (\bar{\nabla}_Y f)X = \eta(Y)AX + \eta(X)AY - 2g(AX, Y)\xi, \tag{3}$$

then, $\bar{M}$ is called a nearly quasi-Sasakian manifold. We have also on nearly quasi-Sasakian manifold $\bar{M}$

$$\bar{\nabla}_X \xi = fAX. \tag{4}$$

Now, let M be a submanifold immersed in $\bar{M}$. The Riemannian metric induced on M is denoted by the same symbol g. Let PM and $P^{\perp}M$ be the Lie algebras of vector fields tangential to M and normal to M respectively and N be the induced Levi-Civita connection on M, then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{5}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{6}$$

for any $X, Y \in PM$ and $V \in P^{\perp}M$, where $\nabla^{\perp}$ is the connection on the normal bundle $P^{\perp}M$, h is the second fundamental form and A_V is the Weingarten map associated with V as

$$g(A_V X, Y) = g(h(X, Y), V). \tag{7}$$

For any $X \in PM$ and $V \in P^{\perp}M$, we write

$$fX = PX + VX \quad (PX \in PM \quad and \quad VX \in P^{\perp}M), \tag{8}$$

$$fV = tV + nV \quad (tV \in PM \quad and \quad nV \in P^{\perp}M). \tag{9}$$

The submanifold M is invariant if V is identically zero. On the other hand, M is anti-invariant if P is identically zero. From (1) and (8), we have

$$g(X, PY) = -g(PX, Y) \tag{10}$$

for any $X, Y \in PM$. If we put $Q = P^2$ we have

$$(\nabla_X Q)Y = \nabla_X QY - Q\nabla_X Y, \tag{11}$$

$$(\nabla_X P)Y = \nabla_X PY - P\nabla_X Y, \tag{12}$$

$$(\nabla_X V)Y = \nabla^{\perp}_X VY - V\nabla_X Y \tag{13}$$

for any $X, Y \in PM$. In view of (5), (8), and (4) it follows that

$$\bar{\nabla}_X \xi = PAX, \tag{14}$$

$$h(X, \xi) = VAX. \tag{15}$$

The mean curvature vector H of M is given by

$$H = \frac{1}{n}\,\mathrm{trace}\,(h) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i), \tag{16}$$

where n is the dimension of M and $e_1, e_2, ..., e_n$ is a local orthonormal frame of M. A submanifold M of an contact metric manifold $\bar{M}$ is said to be totally umbilical if

$$h(X, Y) = g(X, Y)H, \tag{17}$$

where H is the mean curvature vector. A submanifold M is said to be totally geodesic if $h(X, Y) = 0$, for each $X, Y \in \Gamma(PM)$ and M is said to be minimal if $H = 0$.

2. Pseudo-slant submanifolds of nearly quasi-Sasakian manifolds

The purpose of this section is study the existence of pseudo-slant submanifolds of nearly quasi-Sasakian manifolds.

Definition 1. *Let M be a submanifold of a nearly quasi-Sasakian manifold $\bar{M}$. For each non-zero vector X tangent to M at x, the angle $\theta(x) \in [0, \pi/2]$, between fX and PX is called the slant angle or the Wirtinger angle of M . If the slant angle is constant for each $X \in \Gamma(PM)$ and $X \in M$, then the submanifold is also called the slant submanifold. If $\theta = 0$ the submanifold is invariant submanifold. If $\theta = \pi/2$ then it is called anti-invariant submanifold. If $\theta(x) \in [0, \pi/2]$, then it is called proper-slant submanifold.*

Now, we will give the definition of pseudo-slant submanifold which are a generalization of the slant submanifolds.

Definition 2. *We say that M is a pseudo-slant submanifold of nearly quasi Sasakian manifold $\bar{M}$ if there exist two orthogonal distributions $D^{\perp}$ and D_{θ} on M such that*

1) *PM admits the orthogonal direct decomposition $PM = D^{\perp} \oplus D_{\theta}$, $\xi = \Gamma(D)$.*
2) *The distribution $D^{\perp}$ is anti-invariant i.e., $f(D^{\perp}) \subseteq P^{\perp}M$.*
3) *The distribution D_{θ} is a slant with slant angle $\theta \neq 0$, that is, the angle between $f(D_{\theta})$ and D_{θ} is a constant.*

From the definition, it is clear that if $\theta = 0$, then the pseudo-slant submanifold is a semi invariant submanifold. On the other hand, if $\theta = \pi/2$, submanifold becomes an anti-invariant.

On the other hand we suppose that M is a pseudo-slant submanifold of nearly quasi Sasakian manifold $\bar{M}$ and we denote the dimensions of distributions $D^{\perp}$ and D_{θ} by d_1 and d_2, respectively, then we have the following cases:

1) If $d_2 = 0$ then M is an anti-invariant submanifold.
2) If $d_1 = 0$ and $\theta = 0$, then M is an invariant submanifold.
3) If $d_1 = 0$ and $\theta \neq 0$, then M is a proper slant submanifold with slant angle θ.
4) If $d_1.d_2 \neq 0$ and $\theta \in [0, \pi/2]$ then M is a proper pseudo-slant submanifold.

Theorem 1. *Let M be a submanifold of a nearly quasi-Sasakian manifold $\bar{M}$ such that $\xi \in PM$. Then M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$P^2 = -\lambda\{I - \eta \otimes \xi\} \tag{18}$$

Furthermore, in such a case if θ is the slant angle of M then $\lambda = cos^2\theta$.

Corollary 1. *Let M be a slant submanifold of nearly quasi-Sasakian manifold $\bar{M}$ with slant angle θ. Then for any $X, Y \in \Gamma(PM)$ we have*

$$g(PX, PY) = cos^2\theta(g(X, Y) - \eta(X)\eta(Y)), \tag{19}$$

$$g(VX, VY) = sin^2\theta(g(X, Y) - \eta(X)\eta(Y)). \tag{20}$$

Let M a proper pseudo slant submanifold of a contact metric manifold $\bar{M}$ and the projections on $D^{\perp}$ and D_{θ} by P_1 and P_2, respectively, then for any vector field $X \in \Gamma(PM)$, we can write

$$X = P_1X + P_2X + \eta(X)\xi. \tag{21}$$

Now applying f on both sides of equation (3.4), we obtain

$$fX = fP_1X + fP_2X$$

that is,

$$PX + VX = VP_1X + PP_2X + VP_2X. \tag{22}$$

We can easily to see

$$PX = PP_2X, \quad VX = VP_1X + VP_2X \quad \text{and} \tag{23}$$

$$fP_1X = VP_1X, \quad TP_1X = 0, \quad fP_2X = TP_2X + VP_2X, \tag{24}$$

$$TP_2X \in \Gamma(D_{\theta}). \tag{25}$$

If we denote the orthogonal complementary of fPM in $D^{\perp}M$ by μ, then the normal bundle $P^{\perp}M$ can be decomposed as follows

$$P^{\perp}M = V(D^{\perp}) \oplus V(D_{\theta}) \oplus \mu, \tag{26}$$

where μ is an invariant sub bundle of $P^{\perp}M$ as $N(D^{\perp})$ and $N(D_{\theta})$ are orthogonal distribution on M. Indeed, $g(Z, X) = 0$ for each $Z \in \Gamma(D^{\perp})$ and $X \in \Gamma(D_{\theta})$. Thus, by equation (1) and (25), we can write

$$g(VZ, VX) = g(fZ, fX) = g(Z, X) = 0, \tag{27}$$

that is, the distributions $V(D^{\perp})$ and $V(D_{\theta})$ are mutually perpendicular. In fact, the decomposition (26) is an orthogonal direct decomposition.

3. Integrability of Distributions

In this section we shall discuss the integrability of involved distributions.

Theorem 2. *Let M be a pseudo-slant submanifold of nearly quasi Sasakian manifold $\bar{M}$. Then for all $X, Y \in D^{\perp}$ we have*

$$\begin{gathered} A_{fY}X - A_{fX}Y = \nabla_X(PY) + h(X, PY) - A_{VY}X + \nabla^{\perp}_X(VY) - \\ -P(\nabla_X Y) - V(\nabla_X Y) - V(h(X, Y)). \end{gathered} \tag{28}$$

Proof. In view of (7), we get

$$g(A_{fY}X, Z) = g(h(X, Z), fY) = -g(fh(X, Z), Y). \tag{29}$$

From (5) and (29), we get

$$\begin{gathered} g(A_{fY}X, Z) = -g(f\bar{\nabla}_Z X, Y) + g(f\nabla_Z X, Y) \\ = -g(f\bar{\nabla}_Z X, Y) \quad \text{since} \quad f\nabla_Z X \in P^{\perp}M \\ = g((\bar{\nabla}_Z f)X, Y) - g(\bar{\nabla}_Z fX, Y). \end{gathered} \tag{30}$$

Now, for $X \in D_1$, $fX \in P^{\perp}M$. Hence, from (6) we have

$$\bar{\nabla}_Z fX = -A_{fX}Z + \nabla^{\perp}_Z fX. \tag{31}$$

Combining (30) and (31), we obtain

$$g(A_{fY}X, Z) = g((\bar{\nabla}_Z f)X, Y) + g(A_{fX}Z, Y). \tag{32}$$

Since $h(X, Y) = h(Y, X)$, if follows from (7) that

$$g(A_{fX}Z, Y) = g(A_{fX}Y, Z).$$

Hence, from (32) we obtain, with the help of (3),

$$\begin{gathered} g(A_{fY}X, Z) - g(A_{fX}Y, Z) = \eta(Z)g(AX, Y) + \eta(X)g(AZ, Y) - \\ -2g(AZ, X)\eta(Y) + g((\bar{\nabla}_X f)Y, Z) = \\ = \eta(Z)g(AX, Y) + \eta(X)g(AZ, Y) - 2g(AZ, X)\eta(Y) + \\ + g(\bar{\nabla}_X(PY) + \bar{\nabla}_X(PY) - f(\bar{\nabla}_X Y) - f(h(X, Y)), Z) = \\ = \eta(Z)g(AX, Y) + \eta(X)g(AZ, Y) - 2\eta(Y)g(AZ, X) + \\ + g(\nabla_X(PY) + h(X, PY) - A_{VY}(X) + \bar{\nabla}^{\perp}_X(VY) - \\ -P(\nabla_X Y) - V(\nabla_X Y) - P(h(X, Y) - V(h(X, Y))), Z). \end{gathered} \tag{33}$$

Since $X, Y, Z \in D^{\perp}$ an orthonormal distribution to the distribution $\langle \xi \rangle$ it follows that $\eta(X) = \eta(Y) = 0$. Therefore, the above equation reduces to

$$A_{fY}X - A_{fX}Y = \nabla_X(PY) + h(X, PY) - A_{VY}X + \nabla^{\perp}_X(VY) - \\ - P(\nabla_X Y) - V(\nabla_X Y) - V(h(X, Y)).$$

Theorem 3. *In a pseudo-slant submanifold of a nearly quasi Sasakian manifold is given by*

$$(\nabla_X P)Y = A_{VY}X + A_{VX}Y + th(X, Y) + \eta(Y)AX + \eta(X)AY - \\ - 2g(AX, Y)\xi + \nabla_Y(PX) + P(\nabla_Y X) + P(h(Y, X)). \tag{34}$$

Proof. Let $X, Y \in PM$, we have

$$\bar{\nabla}_X fY = (\bar{\nabla}_X f)Y + f(\bar{\nabla}_X Y)$$

From (8) and (9), we obtain

$$\bar{\nabla}_X PY + \bar{\nabla}_X VY = (\bar{\nabla}_X f)Y + f(\nabla_X Y + h(X, Y))$$

Also from (8) and (9), we obtain

$$\bar{\nabla}_X PY + \bar{\nabla}_X VY = (\bar{\nabla}_X f)Y + P(\nabla_X Y) + V(\nabla_X Y) + th(X, Y) + nh(X, Y)$$

Using (5) and (6) from above, we obtain

$$\nabla_X PY + h(X, PY) - A_{VY}X + \nabla^{\perp}_X VY = \eta(Y)AX + \eta(Y)AX - \\ - 2g(AX, Y)\xi + P(\nabla_X Y) + V(\nabla_X Y) + th(X, Y) + nh(X, Y) - \\ -\nabla_Y PX - h(Y, PX) + A_{VX}Y - \nabla^{\perp}_Y VX + P\nabla_Y X + \\ +V\nabla_Y X + P(h(Y, X)) + V(h(Y, X)). \tag{35}$$

Comparing tangential and normal parts, we obtain

$$\nabla_X PY - A_{VY}X = \eta(Y)AX + \eta(X)AY - 2g(AX, Y)\xi + P(\nabla_X Y) + \\ +th(X, Y) + \nabla_Y(PX) + A_{VX}Y + P(\nabla_Y X) + P(h(Y, X)). \tag{36}$$

That is,

$$(\nabla_X P)Y = A_{VY}X + A_{VX}Y + th(X, Y) + \eta(Y)AX + \eta(X)AY - \\ - 2g(AX, Y)\xi + \nabla_Y(PX) + P(\nabla_Y X) + P(h(Y, X)). \tag{37}$$

Theorem 4. *Let M be a pseudo-slant of a nearly quasi Sasakian manifold $\bar{M}$. Then the anti-invariant distribution $D^{\perp}$ is integrable if and only if for any $Z, W \in \Gamma(D^{\perp})$.*

$$A_{VW}Z + A_{VZ}W + 2T\nabla_Z W + 2th(W, Z) = -\eta(W)AZ - \eta(Z)AW + 2g(AZ, W)\xi. \tag{38}$$

Proof. Let $Z, W \in \Gamma(D^{\perp})$ and using (3), we obtain

$$(\bar{\nabla}_Z f)W + (\bar{\nabla}_W f)Z = \eta(W)AZ + \eta(Z)AW - 2g(AZ, W)\xi,$$

which is equivalent to

$$\bar{\nabla}_Z fW - f\bar{\nabla}_Z W + \bar{\nabla}_W fW - f\bar{\nabla}_W Z = \eta(W)AZ + \eta(Z)AW - 2g(AZ, W)\xi.$$

By using (5), (6), (8) and (9), we obtain

$$\eta(W)AZ + \eta(Z)AW - 2g(AZ, W)\xi = \bar{\nabla}_Z NW - T\nabla_Z W - V\nabla_Z W - th(W, Z) - \\ - nh(W, Z) + \bar{\nabla}_W VZ - T\nabla_W Z - V\nabla_W Z - th(W, Z) - nh(W, Z).$$

So we have

$$\eta(W)AZ + \eta(Z)AW - 2g(AZ, W)\xi = -A_{VW}Z + \nabla^{\perp}_Z VW - T\nabla_Z W - V\nabla_Z W - \\ - 2th(W, Z) - A_{VZ}W + \nabla^{\perp}_W VZ - T\nabla_W Z - V\nabla_W Z - 2nh(W, Z).$$

Corresponding the tangent components of the last equation, we conclude

$$-\eta(W)AZ - \eta(Z)AW + 2g(AZ, W)\xi = A_{VW}Z + T\nabla_Z W + 2th(W, Z) + A_{VZ}W + T\nabla_W Z.$$

From the above equation, we can infer

$$-\eta(W)AZ - \eta(Z)AW + 2g(AZ, W)\xi = A_{VW}Z + A_{VZ}W + 2T\nabla_Z W - \\ -T(\nabla_Z W - \nabla_W Z) + 2th(W, Z) \\ T[Z, W] = A_{VW}Z + A_{VZ}W + 2T\nabla_Z W + 2th(W, Z) + \\ +\eta(W)A + \eta(Z)AW - 2g(AZ, W)\xi$$

Thus $[Z, W] \in \Gamma(D^{\perp})$ if and only if (38) is satisfied.

Theorem 5. *Let M be a pseudo-slant submanifold of a nearly quasi Sasakian manifold $\bar{M}$. Then the slant distribution D_θ is integrable if and only if for any $X, Y \in \Gamma(D_\theta)$*

$$P_1\{\nabla_X PY - P\nabla_Y X + (\nabla_Y P)X - A_{VX}Y - A_{VY}X - 2th(X, Y) - \\ - \eta(Y)AX - \eta(X)AY + 2g(AX, Y)\xi\} = 0. \quad (39)$$

Proof. For any $X, Y \in \Gamma(D_\theta)$ and we denote the projections on $D^{\perp}$ and D_θ by P_1 and P_2 , respectively, then for any vector fields $X, Y \in \Gamma(D_\theta)$, by using equation (4), we obtain

$$(\bar{\nabla}_X f)Y + (\bar{\nabla}_Y f)X = \eta(Y)AX + \eta(X)AY - 2g(AX, Y)\xi$$

or

$$\bar{\nabla}_X fY - f\bar{\nabla}_X Y + \bar{\nabla}_Y fX - f\bar{\nabla}_Y X = \eta(Y)AX + \eta(X)AY - 2g(AX, Y)\xi.$$

By using equations (5), (6), (8), and (9), we can write

$$\bar{\nabla}_X PY + \bar{\nabla}_X VY - f(\nabla_X Y + h(X, Y)) + \bar{\nabla}_Y PX + \bar{\nabla}_Y VX - \\ -f(\nabla_Y X + h(X, Y)) = \eta(Y)AX + \eta(X)AY - 2g(AX, Y)\xi + \\ + \nabla_X PY + h(X, PY) - A_{VY}X + \nabla^{\perp}_X VY - P\nabla_X Y - \\ -V\nabla_X Y - th(X, Y) - nh(X, Y) + \nabla_Y PX + h(Y, PX) - \\ - A_{NX}Y + \nabla^{\perp}_Y VX - P\nabla_Y X - V\nabla_Y X - th(X, Y) - \\ - nh(X, Y) = \eta(Y)AX + \eta(X)AY - 2g(AX, Y)\xi, \quad (40)$$

From tangential components of (40) reach

$$\begin{aligned}\nabla_X PY - P\nabla_X Y + (\nabla_Y P)X - A_V XY - A_V YX - 2th(X,Y) = \\ = \eta(Y)AX + \eta(X)AY - 2g(AX,Y)\xi,\end{aligned} \tag{41}$$

$$\begin{aligned}P[X,Y] = \nabla_X PY - P\nabla_X Y + (\nabla_Y P)X - A_V XY - \\ -A_V YX - 2th(X,Y) - \eta(Y)AX - \eta(X)AY + 2g(AX,Y)\xi.\end{aligned} \tag{42}$$

Applying P_1 to (42), we get (39).

Theorem 6. *Let M be a pseudo-slant submanifold of a nearly quasi Sasakian manifold $\bar{M}$. Then the distribution $D^\perp \oplus \xi$ is integrable if and only if for any $Z, W \in \Gamma(D^\perp \oplus \xi)$*

$$A_{fZ}W - A_{fW}Z = \frac{1}{3}[\eta(AZ)W - \eta(Z)AW + \eta(W)AZ - \eta(AW)Z].$$

Proof. For any $Z, W \in \Gamma(D^\perp \oplus \xi)$ and $U \in \Gamma(PM)$, by using (7), we can write

$$2g(A_{fZ}W, U) = g(h(U,W), fZ) + g(h(U,W), fZ).$$

By using (5), we have

$$\begin{aligned}2g(A_{fZ}W, U) = g(\bar{\nabla}_W U, fZ) + g(\bar{\nabla}_U W, fZ) = \\ = -g(f\bar{\nabla}_W U, Z) - g(f\bar{\nabla}_U W, Z).\end{aligned}$$

So we have

$$\begin{aligned}2g(A_{fZ}W, U) = g((\bar{\nabla}_W f)U + (\bar{\nabla}_U f)W, Z) - \\ - g(\bar{\nabla}_W fU, Z) - g(\bar{\nabla}_U fW, Z).\end{aligned}$$

By using equation (3), we obtain

$$\begin{aligned}2g(A_{fZ}W, U) = g(\eta(U)AW + \eta(W)AU - 2g(AW,U)\xi, Z) - g(\bar{\nabla}_W fU, Z) - g(\bar{\nabla}_U fW, Z) = \\ = g(\eta(U)AW + \eta(W)AU - 2g(AW,U)\xi, Z) - g(\bar{\nabla}_W Z, fU) - g(-A_{fW}U, Z) = \\ = g(\eta(AW)Z + \eta(W)AZ - 2\eta(AW)Z, U) - g(f\bar{\nabla}WZ, U) + g(A_{fW}U, Z) = \\ = g(\eta(AW)Z + \eta(W)AZ - 2\eta(AW)Z - P\nabla_W Z - th(Z,W), U) + g(A_{fW}Z, U)\end{aligned}$$

which is equivalent to

$$2A_{fZ}W = \eta(W)AZ - \eta(AW)Z + A_{fW}Z - P\nabla_W Z - th(Z,W). \tag{43}$$

Take $Z = W$ in (43), we infer

$$2A_{fW}Z = \eta(Z)AW - \eta(AZ)W + A_{fZ}W - P\nabla_Z W - th(W,Z). \tag{44}$$

By using equation (43) and (45), we obtain

$$3(A_{fZ}W - A_{fW}Z) = P[Z,W] - \eta(Z)AW + \eta(AZ)W + \eta(W)AZ - \eta(AW)Z, \tag{45}$$

thus the distribution $D^\perp \oplus \xi$ is integrable if and only if $P[Z,W] = 0$ which proves our assertion.

4. Totally umbilical pseudo-slant submanifolds

In this section we shall consider M as a totally umbilical pseudo-slant submanifold of nearly quasi Sasakian manifold $\bar{M}$. We have the following preparatory results.

Theorem 7. *Let M be a totally umbilical pseudo-slant submanifold of a nearly quasi Sasakian manifold $\bar{M}$. Then at least one of the following statements is true,*

1) $dim(D^{\perp}) = 1$,
2) $H \in \Gamma(\mu)$,
3) *M is proper pseudo-slant submanifold.*

Proof. Let $Z \in \Gamma(D^{\perp}$ and using (3), we obtain

$$(\bar{\nabla}_Z f)Z = \eta(Z)AZ - g(AZ, Z)\xi,$$

$$\bar{\nabla}_Z VZ - f(\bar{\nabla}_Z Z + h(Z, Z)) = \eta(Z)AZ - g(AZ, Z)\xi.$$

From the last equation, we have

$$-A_{VZ}Z + \nabla^{\perp}_Z VZ - N\nabla_Z Z - th(Z, Z) - nh(Z, Z) = \eta(Z)AZ - g(AZ, Z)\xi. \quad (46)$$

From (12) and from the tangential components of (46), we obtain

$$A_{VZ}Z + th(Z, Z) = -\eta(Z)AZ + g(AZ, Z)P\xi. \quad (47)$$

Taking the product by $W \in \Gamma(D^{\perp})$, we obtain

$$g(A_{VZ}Z + th(Z, Z) + \eta(Z)AZ - g(AZ, Z)P\xi, W) = 0.$$

It implies that

$$g(h(Z, W), NZ) + g(th(Z, Z), W) + \eta(Z)g(AZ, W) - g(AZ, Z)g(P\xi, W) = 0. \quad (48)$$

Since M is totally umbilical submanifold, we obtain

$$g(Z, W)g(H, VZ) + g(Z, Z)g(tH, W) + \eta(Z)g(AZ, W) - g(AZ, Z)g(P\xi, W) = 0, \quad (49)$$

that is

$$-g(tH, Z)W + g(tH, W)Z + g(AZ, W)\xi - g(P\xi, W)AZ = 0. \quad (50)$$

Here tH is either zero or Z and W are linearly dependent vector fields. If $tH \neq 0$, then $dim\Gamma(D^{\perp}) = 1$.

Otherwise $H \in \Gamma(\mu)$. Since $D_{\theta} \neq 0$, M is pseudo-slant submanifold. Since $\theta \neq 0$ and $d_1, d_2 \neq 0$, M is proper pseudo-slant submanifold.

Theorem 8. *Let M be totally umbilical proper pseudo-slant submanifold of nearly quasi Sasakian manifold $\bar{M}$. Then M is an either totally geodesic submanifold or it is an anti-invariant if $H, \nabla^{\perp}_X H \in \Gamma(\mu)$.*

Proof. Since the ambient space is nearly quasi Sasakian manifold, for any $X \in \Gamma(PM)$, by using 3, we have

$$(\bar{\nabla}_X f)X = \eta(X)AX - g(AX,X)\xi\bar{\nabla}_X fX - f\bar{\nabla}_X X = \eta(X)AX - g(AX,X)\xi. \quad (51)$$

Using (5), (7), (8) and (12) in (51) and we get

$$\begin{gathered}\nabla_X PX - g(X,PX)H - A_{VX}X + \nabla^{\perp}_X VX = \\ = f\nabla_X X + g(X,X)fH + \eta(X)AX - g(AX,X)\xi.\end{gathered} \quad (52)$$

Applying product fH to the above equation we get

$$g(\nabla^{\perp}_X VX, fH) = g(V\nabla_X X, fH) + g(X,X)\|H\|^2 - g(AX,X)g(V\xi, fH) \quad (53)$$

taking into account (6), we get

$$g(\bar{\nabla}^{\perp}_X VX, fH) = g(X,X)\|H\|^2 - g(AX,X)g(V\xi, fH). \quad (54)$$

Now, for any $X \in \Gamma(PM)$, we obtain

$$\bar{\nabla}_X fH = (\bar{\nabla}_X f)H + f\bar{\nabla}_X H. \quad (55)$$

In view of (6), (8), (9), (17) and (55) we obtain

$$-A_{fH}X + \nabla^{\perp}_X fH = (\bar{\nabla}_X f)H - PA_H X - VA_H X + t\nabla^{\perp}_X H + n\nabla^{\perp}_X H. \quad (56)$$

Applying product VX to the above equation we get

$$g(\nabla^{\perp}_X fH, VX) = g((\bar{\nabla}_X f)H, VX) - g(VA_H X, VX),$$

$$g(\bar{\nabla}_X fH, VX) = g((\nabla_X n)H + h(tH,X) + VA_H X, VX) - g(VA_H X, VX).$$

By using (7), (17) and (20), we have

$$g(\bar{\nabla}_X fH, VX) = -\sin^2\theta\{g(X,X)\|H\|^2 - g(h(X,\xi),H)\eta(X)\}.$$

From (15), we obtain

$$\begin{gathered}g(\bar{\nabla}_X fH, VX) = -\sin^2\theta\{g(X,X)\|H\|^2\}, \\ g(\bar{\nabla}_X VH, fX) = \sin^2\theta\{g(X,X)\|H\|^2\}.\end{gathered} \quad (57)$$

Thus, (54) and (57) imply

$$\begin{gathered}g(X,X)\|H\|^2 = \sin^2\theta\{g(X,X)\|H\|^2\}, \\ \cos^2\theta g(X,X)\|H\|^2 = 0.\end{gathered} \quad (58)$$

From (58), we conclude that $g(X,X)\|H\|^2 = 0$, for any $X \in \Gamma(PM)$. Since M proper pseudo slant submanifold of nearly quasi Sasakian manifold we obtain $H = 0$. This tells us that M is totally geodesic in $\bar{M}$.

Theorem 9. *Let M be totally umbilical proper pseudo-slant submanifold of nearly quasi Sasakian manifold $\bar{M}$. Then at least one of the following statements is true.*

1) $H \in \mu$.
2) $g(\nabla_{PX}\xi, X) = 0$.
3) $\eta((\nabla_X P)X) = 0$.
4) *M is a anti-invariant submanifold.*
5) *If M proper slant submanifold then, $dim(M) \geq 3$, for any $X \in \Gamma(PM)$.*

Proof. From equation (3) and M is nearly quasi Sasakian manifold, we have

$$\bar{\nabla}_X fX - f\bar{\nabla}_X X = \eta(X)AX - g(AX, X)\xi.$$

By using (5), (6), (8) and (9), we have

$$\begin{gathered}\nabla_X PX + h(X, PX) - A_V XX + \nabla^{\perp}_X VX - P\nabla_X X - \\ -V\nabla_X X - th(X, X) - nh(X, X) = \eta(X)AX - g(AX, X)\xi\end{gathered} \tag{59}$$

tangential components of (59), we obtain

$$\nabla_X PX - P\nabla_X X - th(X, X) - A_{VX}X = \eta(X)AX. \tag{60}$$

Since M is a totally umbilical pseudo-slant submanifold, by using (7) and (17), we can write

$$g(A_{VX}X, X) = g(h(X, X), VX) = g(H, VX)g(X, X) = g(g(H, VX)X, X) = 0. \tag{61}$$

If $H \in \Gamma(\mu)$, then from (60), we obtain

$$\nabla_X PX - P\nabla_X X = \eta(X)AX.$$

Taking the product of (61) by ξ, we obtain

$$g(\nabla_X PX, \xi) - \eta(P\nabla_X X) = \eta(X)\eta(AX)g(\nabla_X PX, \xi) = 0. \tag{62}$$

Interchanging X by PX in (62), we derive

$$g(\nabla_{PX}P^2X, \xi) = 0 \quad \Rightarrow \quad g(\nabla_{PX}\xi, P^2X) = 0$$

by using (18), we have

$$g(\nabla_{PX}\xi, -\cos^2\theta(X - \eta(X)\xi) = 0 \quad \Rightarrow \quad \cos^2\theta g(\nabla_{PX}\xi, (X - \eta(X)\xi) = 0.$$

Since, M is a proper pseudo-slant submanifold, we have

$$g(\nabla_{PX}\xi, (X - \eta(X)\xi) = 0.$$

From which

$$g(\nabla_{PX}\xi, X) = \eta(X)g(\nabla_{PX}\xi, \xi). \tag{63}$$

Now, we have $g(\xi, \xi) = 1$. Taking the covariant derivative of above equation with respect to PX for any $X \in \Gamma(PM)$, we obtain $g(\nabla_{PX}\xi, \xi) + g(\xi, \nabla_{PX}\xi) = 0$ which implies $g(\nabla_{PX}\xi, \xi) = 0$ and then (63) gives

$$g(\nabla_{PX}\xi, X) = 0. \tag{64}$$

This proves 2) of theorem.

Now, Inter changing X by PX in the equation (64), we derive

$$g(\nabla_{P^2X}\xi, TX) = g(\nabla_{\cos^2\theta(-X+\eta(X)\xi)}\xi, PX) = 0,$$

$$\cos^2\theta g(\nabla_{(-X+\eta(X)\xi)}\xi, PX) = 0,$$

$$-\cos^2\theta g(\nabla_X\xi, PX) + -\cos^2\theta\eta(X)g(\nabla_\xi\xi, PX) = 0.$$

Since $\nabla_\xi\xi = 0$, we obtain

$$\cos^2\theta g(\nabla_X\xi, PX) = 0. \tag{65}$$

From (65) if $\cos\theta = 0$, $\theta = \pi/2$ then M is an anti-invariant submanifold. On the other hand, $g(\nabla_X\xi, PX) = 0$, that is $\nabla_X\xi = 0$. This implies that ξ is a the Killing vector field on M. If the vector field ξ is not Killing, then we can take at least two linearly independent vectors X and PX to span D_θ, that is, the $dim(M) \geq 3$.

Example 1. Suppose M be a submanifold of R^7 with coordinates $(x_1, x_2, x_3, y_1, y_2, y_3, w)$, defined by

$$x_1 = \sqrt{3}u\sinh\alpha, x_2 = -v\cosh\alpha, x_3 = s\sinh z,$$

$$y_1 = v\cosh\alpha, y_2 = 2v\cosh\alpha, y_3 = -s\sinh z, t = w,$$

where u, v, s and z denote the arbitrary parameters. The tangent bundal space of M is spanned by tangent vectors.

$$e_1 = \sqrt{3}\sinh\alpha\frac{\partial}{\partial x_1}, \quad e_2 = \cosh\alpha\frac{\partial}{\partial y_1} - \cosh\alpha\frac{\partial}{\partial x_2} + 2\cosh\alpha\frac{\partial}{\partial y_2},$$

$$e_3 = \sinh z\frac{\partial}{\partial x_3} - \sinh z\frac{\partial}{\partial y_3}, \quad e_4 = s\cosh z\frac{\partial}{\partial x_3} - s\cosh z\frac{\partial}{\partial y_3}.$$

For the almost contact metric structure ϕ of R^7, choosing

$$\phi(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}, \quad \phi(\frac{\partial}{\partial y_j}) = -\frac{\partial}{\partial x_j}, \quad \phi(\frac{\partial}{\partial t}) = 0, \quad 1 \leqslant i, j \leqslant 3,$$

and $\xi = \frac{\partial}{\partial t}, \eta = dt$. For any vector field $W = \mu_i\frac{\partial}{\partial x_i} + \nu_j\frac{\partial}{\partial x_j} + \lambda\frac{\partial}{\partial w} \in T(R^7)$, then we have

$$\phi Z = \mu_i\frac{\partial}{\partial y_j} - \nu_j\frac{\partial}{\partial x_i}, \quad g(\phi Z, \phi Z) = \mu_i^2 + \nu_j^2,$$

$$g(Z, Z) = \mu_i^2 + \nu_j^2 + \lambda^2, \quad \eta(Z) = g(Z, \xi) = \lambda$$

and

$$\phi^2 Z = -\mu_i\frac{\partial}{\partial x_i} - \nu_j\frac{\partial}{\partial y_i} - \lambda\frac{\partial}{\partial t} + \lambda\frac{\partial}{\partial t} = -Z + \eta(Z)\xi$$

for any $i, j = 1, 2, 3$. It follows that $g(\phi Z, \phi Z) = g(Z, Z) - \eta^2(Z)$. Thus (ϕ, ξ, η, g) is an almost contact metric structure on R^7. Thus we have

$$\phi e_1 = \sqrt{3}\sinh\alpha\frac{\partial}{\partial y_1}, \quad \phi e_2 = -\cosh\alpha\frac{\partial}{\partial x_1} - \cosh\alpha\frac{\partial}{\partial y_2} - 2\cosh\alpha\frac{\partial}{\partial x_2},$$

$$\phi e_3 = \sinh z \frac{\partial}{\partial y_3} + \sinh z \frac{\partial}{\partial x_3}, \quad \phi e_4 = s \cosh z \frac{\partial}{\partial y_3} + s \cosh z \frac{\partial}{\partial x_3}.$$

By direct calculations, we can infer $D_\theta = span\{e_1, e_2\}$ is a slant distribution with slant angle $\theta = \cos^{-1}(\frac{1}{\sqrt{6}})$. Since

$$g(\phi e_3, e_1) = g(\phi e_3, e_2) = g(\phi e_3, e_4) = g(\phi e_3, e_5) = 0,$$
$$g(\phi e_4, e_1) = g(\phi e_4, e_2) = g(\phi e_4, e_3) = g(\phi e_4, e_5) = 0,$$

e_3 and e_4 are orthogonal to M, $D^\perp = span(e_3, e_4)$ is an anti-invariant distribution. Thus M is a 5-dimensional proper pseudo-slant submanifold of R^7 with its usual almost contact metric structure.

REFERENCES

1. Bejancu A. CR-Submanifold of a Kahler Manifold. *I. Proc. Amer. Math. Soc.*, 1978, vol. 69, pp. 135-142.

2. Bejancu A. *Geometry of CR–Submanifolds*. Dordrecht, D. Reidel Pub. Co, 1986. 172 p.

3. Bejancu A., Papaghiuc N. Semi-Invariant Submanifolds of a Sasakian Manifold. *An. St. Univ. Al. I. Cuza, Iasi supl.*, 1981, vol. XVII, no. 1, I-a, pp. 163-170.

4. Blair D.E. The Theory of Quasi Sasakian Structures. *J. Differential Geom.*, 1967, vol. 1, no. 3–4, pp. 331-345.

5. Carriazo A. New Developments in Slant Submanifolds Theory. *Applicable Mathematics in the Golden Age*. New Delhi, Narosa Publishing House, 2002, vol. 394, pp. 339-356.

6. Chen B.Y. *Geometry of Slant Submanifolds*. Leuven, Katholieke Universiteit Leuven, 1990. 126 p.

7. Cabrerizo J.L., Carriazo A., Fernandez L.M., Fernandez M. Slant Submanifolds in Sasakian Manifolds. *Glasgow Mathematical Journal*, 2000, vol. 42, pp. 125-138. DOI: 10.1017/S0017089500010156.

8. Cabrerizo J.L., Carriazo A., Fernandez L.M., Fernandez M. Semi-Slant Submanifolds of a Sasakian Manifold. *Geom. Dedicata*, 1999, vol. 78, iss. 2, pp. 183-19. DOI: 10.1023/A:1005241320631.

9. Cabrerizo J.L., Carriazo A., Fernandez L.M., Fernandez M. Structure on a Slant Submanifold of a Contact Manifold. *Indian Journal of Pure and Applied Mathematics*, 2000, vol. 31, pp. 857-864.

10. Lotta A. Slant Submanifolds in Contact Geometry. *Bull. Math. Soc. Sci. Math. Roum., Nouv. Ser.*, 1996, vol. 39 (87), no. 1/4, pp. 183-198.

11. Khan V.A., Khan M.A. Pseudo-Slant Submanifolds of a Sasakian Manifold. *Indian Journal of Pure and Applied Mathematics*, 2007, vol. 38, pp. 31-42.

12. Papaghiuc N. Semi-Slant Submanifolds of a Kaehlerian Manifold. *An. Stiint. Univ. Al. I. Cuza-Iasi, Ser. Noua, Mat.*, 1994, vol. 40, pp. 55-61.

13. Rahman S., Khan M.S., Horaira A. Contact CR-Warped Product Submanifolds of Nearly Quasi-Sasakian Manifold. *Konuralp Journal of Mathematics*, 2019, vol. 7, iss. 1, pp. 140-145.

О ПСЕВДО-НАКЛОННЫХ ПОДМНОГООБРАЗИЯХ БЛИЗКО КВАЗИ-САСАКИЕВЫХ МНОГООБРАЗИЙ

Шамсур Раман

Кандидат наук, доцент кафедры математики,
Национальный Университет Урду им. Мауланы Азада
shamsur@rediffmail.com
Политехнический Спутник Кампус Дарбханга, 846002 г. Бихар, Индия

Мод Садик Хан

Кандидат наук, профессор кафедры математики,
Национальный колледж им. Н. Шибли
msadiqkhan.snc@gmail.com
Уттар-Прадеш, 276001 г. Азамгарх, Индия

Абу Хорайра

Научный Сотрудник кафедры математики,
Национальный колледж им. Н. Шибли
aboohoraria@gmail.com
Уттар-Прадеш, 276001 г. Азамгарх, Индия

Аннотация. В работе изучается геометрия псевдо-наклонных подмногообразий близко квази-сасакиевых многообразий. Доказано, что вполне омбилическое правильно наклонное многообразие близко квази-сасакиева многообразия является вполне геодезическим, если вектор средней кривизны $H \in \mu$. Также получены условия интегрируемости распределения псевдо-наклонных подмногообразий близко квази-сасакиевых многообразий.

Ключевые слова: близко квази-сасакиевы многообразия, наклонные многообразия, правильные наклонные подмногообразия, псевдо-наклонные подмногообразия.