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LOG-SOBOLEV INEQUALITIES ON GRAPHS WITH POSITIVE CURVATURE [1]

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Abstract. Based on a global estimate of the heat kernel, some important inequalities such as Poincaré inequality and log-Sobolev inequality, furthermore a tight logarithm Sobolev inequality are presented on graphs, just under a positive curvature condition $CDE'(n,K)$ with some $K > 0$. As consequences, we obtain exponential integrability of integrable Lipschitz functions and moment bounds at the same assumption on graphs.

Key words: Log-Sobolev inequality, Laplacian, $CDE'(n,K)$.

1. Notations and main results

Let $G = (V, E)$ be a symmetric weighted, connected and locally finite graph. The weight function denote by ω_{xy}, we assume $\omega_{xy} = \omega_{yx}$ (symmetry), and $m(x) := \sum_{y\sim x} \omega_{xy} < \infty$, for any $x \in V$ (locally finite). Moreover we assume

$$\omega_{\min} = \inf_{\substack{y \sim x \\ x,\, y \in V}} \omega_{xy} > 0.$$

Given a positive measure $\mu : V \to \mathbb{R}^+$ on graph, and assume that

$$D_\mu := \max_{x\in V} \frac{m(x)}{\mu(x)} < \infty,$$

$$D_\omega := \max_{\substack{y \sim x \\ x,\, y \in V}} \frac{\mu(x)}{\omega_{xy}} < \infty.$$

We denote by $V^{\mathbb{R}}$ the space of real functions on V, by $\ell^p(V, \mu) = \{f \in V^{\mathbb{R}} : \sum_{x\in V} \mu(x)|f(x)|^p < \infty\}, 1 \le p < \infty$, the space of ℓ^p integrable functions on V with respect to the measure μ. For $p = \infty$, let $\ell^\infty(V, \mu) = \{f \in V^{\mathbb{R}} : \sup_{x\in V} |f(x)| < \infty\}$ be the set of bounded functions. If for any $f, g \in \ell^2(V, \mu)$, let the inner product as $\langle f, g\rangle = \sum_{x\in V} \mu(x)f(x)g(x)$, then the space of $\ell^2(V, \mu)$ is a Hilbert space. For every function $f \in \ell^p(V, \mu), 1 \le p < \infty$, we can define the norm as follows

$$\|f\|_p = \left(\sum_{x\in V} \mu(x)|f(x)|^p\right)^{\frac{1}{p}}, 1 \le p < \infty.$$

We denote by $C_0(V) \subset \ell^p(V)$ the dense subset of functions $f : V \to \mathbb{R}$ with finite support. For any graph, it associated with a Dirichlet form, see [7],

$$\begin{aligned} Q^{(D)} :&D(Q) \times D(Q) \to \mathbb{R} \\ &(f, g) \mapsto \frac{1}{2} \sum_{x\sim y} \omega_{xy}(f(y) - f(x))(g(y) - g(x)), \end{aligned}$$

where the form domain $D(Q)$ is defined as the completion of $C_0(V)$ under the norm $\|\cdot\|_Q$ given by

$$\|f\|_Q^2 = \|f\|^2_{\ell^2(V,\mu)} + \frac{1}{2} \sum_{x,y} \omega_{xy}(f(y) - f(x))^2, \forall f \in C_0(V),$$

see Keller and Lenz [9]. For the Dirichlet form $Q^{(D)}$, its infinitesimal generator Δ is called the discrete Laplacian Δ on G by, for any $x \in V$,

$$\Delta f(x) = \frac{1}{\mu(x)} \sum_{y\sim x} \omega_{xy}(f(y) - f(x)),$$

it is a bounded operator because of the assumption of D_μ. And the associated semigroup $P_t : \ell^2(V, \mu) \to \ell^2(V, \mu)$, for any $x \in V$,

$$P_t f(x) = \sum_{y\in V} \mu(y)p(t, x, y)f(y),$$

where $p(t,x,y)$ is the fundamental solution of the heat equation (heat kernel), we refer from [9].

Now we introduce the notion of the CDE' inequality on graphs from [8]. First we need to recall the definition of two bilinear forms associated to the μ-Laplacian.

Definition 1.1. *The gradient form* Γ *and the iterated gradient form* Γ_2 *are respectively defined by, for any* $f, g \in V^{\mathbb{R}}$,

$$2\Gamma(f,g)(x) = \frac{1}{\mu(x)} \sum_{y\sim x} \omega_{xy}(f(y)-f(x))(g(y)-g(x)),$$

and

$$2\Gamma_2(f,g) = \Delta\Gamma(f,g) - \Gamma(f,\Delta g) - \Gamma(\Delta f, g).$$

We write $\Gamma(f) = \Gamma(f,f)$, *and* $\Gamma_2(f) = \Gamma_2(f,f)$.

Definition 1.2. *We say that a graph* G *satisfies the exponential curvature dimension inequality* $CDE(x,n,K)$ *if for any positive function* $f : V \to \mathbb{R}^+$ *such that* $\Delta f(x) < 0$, *we have*

$$\widetilde{\Gamma_2}(f)(x) = \Gamma_2(f)(x) - \Gamma\left(f, \frac{\Gamma(f)}{f}\right)(x) \geq \frac{1}{n}(\Delta f)(x)^2 + K\Gamma(f)(x).$$

We say that $\mathbf{CDE(n,K)}$ *is satisfied if* $CDE(x,n,K)$ *is satisfied for all* $x \in V$.

We say that a graph G *satisfies the* $CDE'(x,n,K)$, *if for any positive function* $f : V \to \mathbb{R}^+$, *we have*

$$\widetilde{\Gamma_2}(f)(x) \geq \frac{1}{n} f(x)^2 \left(\Delta \log f\right)(x)^2 + K\Gamma(f)(x).$$

We say that $\mathbf{CDE'(n,K)}$ *is satisfied if* $CDE'(x,n,K)$ *holds for all* $x \in V$.

We write $\int_V f d\mu = \sum_{x\in V} \mu(x) f(x)$. Now we introduce Poincaré inequality and log-Sobolev inequality on graphs. For all integrable functions f, let

$$V(f) = \int_V f^2 d\mu - \left(\int_V f d\mu\right)^2,$$

be the variance of the function, and for any positive integrable function f such that $\int_V f|\log f| d\mu < \infty$,

$$E(f) = \int_V f \log f d\mu - \int_V f d\mu \log \int_V f d\mu,$$

be the entropy.

A graph $G = (V,E)$ is said to satisfy a **Poincaré inequality** with constant $C > 0$, for any $f \in D(Q)$, if

$$V(f) \leq C \int_V \Gamma(f) d\mu, \qquad \mathbf{(P(C))}$$

a **log-Sobolev inequality** with constants $C > 0, D \geq 0$, for any $f \in D(Q)$, if

$$E(f^2) \leq 2C \int_V \Gamma(f) d\mu + D \int_V f^2 d\mu. \qquad \mathbf{(LS(C,D))}$$

There are a useful fact we will use later, that is, we just examine whether the log-Sobolev inequality holds for any positive value function, since

$$E(f^2) = E(|f|^2) \leq 2C \int_V \Gamma(|f|)d\mu + D \int_V |f|^2 d\mu \leq 2C \int_V \Gamma(f)d\mu + D \int_V f^2 d\mu,$$

since $||a| - |b|| \leq |a - b|$ for any $a, b \in \mathbb{R}$, then $\Gamma(|f|) \leq \Gamma(f)$. When $D = 0$, the logarithmic Sobolev inequality will be called *tight*, denoted by $\mathbf{LS(C)}$.

The main results we derived in this paper are log-Sobolev inequality and the tight log-Sobolev inequality with appropriate constants on graphs, just under a condition with a positive curvature.

Theorem 1.1. *Let $G = (V, E)$ be a locally finite, connected graph satisfying $CDE'(n, K)$ for any $K > 0$, then for any $t_0 > 0$, such that the graph satisfies log-Sobolev inequality $LS(C, D)$, with*

$$C = 2t_0, \qquad D = 2\log M,$$

where $M = M(n, K) = \frac{1}{\left(1 - e^{-\frac{2K}{3}t_0}\right)^n}$.

Theorem 1.2. *Under the same conditions of the above theorem, then the graph satisfies a tight log-Sobolev inequality $LS(C'')$ with*

$$C'' = \frac{3}{K}\left(\left(1 + \frac{nKC'}{3}\right)\log\left(1 + \frac{nKC'}{3}\right) - \frac{nKC'}{3}\log\frac{nKC'}{3}\right) + C',$$

where $C' = 1536 D_\mu D_\omega \pi \frac{n(5n+1)}{K}$.

We also have two applications of the above theorem, including exponential integrability and moment bounds, as follows.

Theorem 1.3. *Let $G = (V, E)$ be a locally finite, connected graph satisfying $CDE'(n, K)$ for any $K > 0$, then there exist a constant $0 < C = C''(n, K) < \infty$ such that, if f is 1-Lipschitz and $\int f d\mu < \infty$, we have*

$$\int_V e^{sf} d\mu \leq e^{s\int_V f d\mu + Cs^2}, \tag{1.1}$$

and

$$\int_V e^{\frac{\sigma^2}{2}f^2} d\mu \leq \frac{1}{\sqrt{1 - 2C\sigma^2}} e^{\frac{\sigma^2(\int_V f d\mu)^2}{2(1-2C\sigma^2)}} < \infty,$$

for every $\sigma^2 < \frac{1}{2C}$.

Theorem 1.4. *Under the same conditions of the above theorem, then there exist a constant $0 < C = C''(n, K) < \infty$ such that for every $p \geq 2$ and every $f \in \ell^p(V, \mu)$, we have*

$$\|f\|_p^2 \leq \|f\|_2^2 + 2C(p-2)\left(\int_V \Gamma(f)^{\frac{p}{2}} d\mu\right)^{\frac{2}{p}}.$$

If f is Lipschitz, then

$$\|f\|_p^2 \leq \|f\|_2^2 + 2C(p-2)\|f\|_{Lip}^2.$$

2. log-Sobolev inequality

In this section we focus on log-Sobolev inequality and tight log-Sobolev inequality with the assumption of positive curvature.

Proof of Theorem 1.1 We divide the proof into several steps.

Step 1. *For any function* $0 < f \in D(Q)$ *and every* $t \geq 0$*,*

$$\int_V f^2 \log f^2 d\mu \leq 2t \int_V \Gamma(f) d\mu + \int_V f^2 \log (P_t f)^2 d\mu. \tag{2.1}$$

This is a relationship between entropy and energy along the semigroup from [C] without diffusion property. We consider a functional $\Phi(s) = \int_V f_k^2 \log P_s f d\mu$, for any $s \geq 0$, and $0 < f \in D(Q)$, where $f_k = \eta_k f$, and η_k is a nondecreasing sequence of finitely supported functions $\{\eta_k\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} \eta_k = \mathbf{1}.$$

Then, since $\omega_{xy} = \omega_{yx}$, and by the Green's formula,

$$\Phi'(s) = \int_V f_k^2 \frac{\Delta P_s f}{P_s f} d\mu = - \int_V \Gamma(\frac{f_k^2}{P_s f}, P_s f) d\mu,$$

and

$$\begin{aligned}
\int_V \Gamma(\frac{f_k^2}{P_s f}, P_s f) d\mu &= \frac{1}{2} \sum_{x\in V} \sum_{y\sim x} \omega_{xy} \left(\frac{f_k^2(y)}{P_s f(y)} - \frac{f_k^2(x)}{P_s f(x)} \right) (P_s f(y) - P_s f(x)) = \\
&= \frac{1}{2} \sum_{x,y\in V} \omega_{xy} \left(f_k^2(y) + f_k^2(x) - \frac{f_k^2(y) P_s f(x)}{P_s f(y)} - \frac{f_k^2(x) P_s f(y)}{P_s f(x)} \right) \leq \\
&\leq \frac{1}{2} \sum_{x,y\in V} \omega_{xy} \left(f_k^2(y) + f_k^2(x) - 2 f_k(x) f_k(y) \right) = \\
&= \int_V \Gamma(f_k) d\mu.
\end{aligned}$$

So that $\Phi'(s) \geq - \int_V \Gamma(f_k) d\mu$ for every $s \geq 0$. Integrating from 0 to t yields

$$\int_V f_k^2 \log f^2 d\mu \leq 2t \int_V \Gamma(f_k) d\mu + \int_V f_k^2 \log (P_t f)^2 d\mu.$$

We have

$$\begin{aligned}
\int_V \Gamma(f_k) d\mu &= \frac{1}{2} \sum_{x,y\in V} \omega_{xy} (f_k(y) - f_k(x))^2 \leq \\
&\leq \frac{1}{2} \sum_{x,y\in V} \omega_{xy} (f_k(y) - f(y))^2 + \frac{1}{2} \sum_{x,y\in V} \omega_{xy} (f(y) - f(x))^2 + \\
&+ \frac{1}{2} \sum_{x,y\in V} \omega_{xy} (f(x) - f_k(x))^2 = \sum_{x\in V} m(x) f(x)^2 (\eta_k(x) - 1)^2 + \int_V \Gamma(f) d\mu \leq \\
&\leq \int_V \Gamma(f) d\mu + D_\mu \sum_{x\in V} \mu(x) f(x)^2 (\eta_k(x) - 1)^2,
\end{aligned}$$

let $k \to \infty$,

$$\sum_{x \in V} \mu(x) f(x)^2 (\eta_k(x) - 1)^2 \to 0,$$

and $\log f, \log P_t f \in \ell^1(V, \mu)$, so we derive what we desire.

Step 2. *If for some $t_0 > 0$, there exists a constant $M > 0$ such that for all $0 \le f \in D(Q)$,*

$$\|P_{t_0} f\|_\infty \le M \|f\|_2,$$

then a logarithmic Sobolev inequality $LS(C, D)$ holds, with

$$C = 2t_0, \qquad D = 2 \log M.$$

When $\|f\|_2 = 1$, since for any $x \in V$, $|P_{t_0} f(x)| \le \|P_{t_0} f\|_\infty$, we have from (2.1),

$$\int_V f^2 \log f^2 d\mu \le 2t \int_V \Gamma(f) d\mu + 2 \log(\|P_{t_0} f\|_\infty) \le 2t \int_V \Gamma(f) d\mu + 2 \log M.$$

If $\|f\|_2 \ne 1$, we can apply $\widetilde{f} = \frac{f}{\|f\|_2}$ to the above inequality, which immediately leads to the above result.

Step 3. *We obtain $LS(C, D)$ on graph just under the assumption of the positive curvature.*

We first introduce the global estimate of heat kernel. If $G = (V, E)$ be a locally finite, connected graph satisfying $CDE'(n, K)$ with $K > 0$, then the measure μ is finite (see [6]). We may then assume μ is probability measure, then $\lim_{t \to \infty} p(t, x, \cdot) = 1$ in the case of probability measure. Under the same condition, for any $x, y \in V$, $t > 0$,

$$p(t, x, y) \le \frac{1}{\left(1 - e^{-\frac{2K}{3} t}\right)^n}. \tag{2.2}$$

(see [8, Proposition 7.5]).

Then, by Holdër inequality and combining with (2.2), we obtain for some $t_0 > 0$,

$$\|P_{t_0} f\|_\infty \le M \|f\|_2,$$

where $M = \frac{1}{\left(1 - e^{-\frac{2K}{3} t_0}\right)^n}$. we together with step 2 immediately end the proof.

Proof of Theorem 1.2 From [5], we can divide the proof of the tight log-Sobolev inequality at the assumption of curvature into several steps.

Step 1. *A logarithmic Sobolev inequality $LS(C, D)$ together with Poincaré inequality $P(C')$ implies a tight logarithmic Sobolev inequality $LS(C + C'(\frac{D}{2} + 1))$.*

We first introduce Rothaus's Lemma in the discrete condition from [10]. If $f \in V^{\mathbb{R}}$ such that $\int_V f^2 \log(1 + f^2) d\mu < \infty$, then for every $a \in \mathbb{R}$,

$$E((f + a)^2) \le E(f^2) + 2 \int_V f^2 d\mu.$$

Applied to $\hat{f} = f - \int_V f d\mu$ with $a = \int_V f d\mu$ of the above inequality, yields

$$E(f^2) \le E(\hat{f}^2) + 2 \int_V \hat{f}^2 d\mu,$$

by the logarithmic Sobolev inequality $LS(C, D)$ applied to $\hat{f}$, and combining to the above inequality we get,

$$E(f^2) \leq 2C \int_V \Gamma(f) d\mu + (D+2) \int_V \hat{f}^2 d\mu,$$

and since $\int_V \hat{f}^2 d\mu = \int_V f^2 d\mu - \left(\int_V f d\mu\right)^2$ for the probability measure μ (the finiteness of measure is true when the graph satisfies $CDE'(n, K)$ with $K > 0$, see [8]), it remains to use the Poincaré inequality $P(C')$, the conclusion is therefore established.

Before the next step, we first show two theorems from [8] and [2]. It will be useful to prove the graph also satisfy a Poincaré inequality when the curvature of graph is bounded by a positive number.

Lemma 2.1. *If a graph be a locally finite, connected, and satisfy $CDE'(n, K)$ with $K > 0$, then the diameter of the natural distance on the graph is finite, moreover the upper bound quantitative estimation is*

$$D \leq 2\pi \sqrt{\frac{6 D_\mu n}{K}}.$$

We refer a lower bound estimate of eigenvalues from [2].

Lemma 2.2. *Let a finite graph Ω satisfy $CDE(n, \rho)$ with $\rho \geq 0$, then*

$$\lambda_1 \geq \frac{1}{64(5n+1) D_\omega D^2}.$$

Since $CDE'(n, K)$ implies $CDE(n, K)$, see [3], so we have the same result under the condition of $CDE'(n, K)$ with $K \geq 0$. From Lemma 2.1, we find the graph is finite when it satisfies $CDE'(n, K)$ with $K > 0$. So we can use the above lemma when the curvature is bounded by a positive number.

Step 2. *If a graph be a locally finite, connected, and satisfies $CDE'(n, K)$ with $K > 0$, then the graph satisfies a spectral inequality $P(C')$ with*

$$C' = C'(n, K) = 1536 D_\mu D_\omega \pi \frac{n(5n+1)}{K} < \infty.$$

For any $f \in D(Q)$, from spectral theory on graph G, let λ_i be the i-th eigenvalue, and $\{\psi_i\}_{i=0}^{N-1}$ (N is the number of vertices in the subgraph Ω) be an orthonormal basis of eigenfunctions, i.e.

$$\Delta \psi_i = \lambda_i \psi_i,$$

and

$$\langle \psi_i, \psi_j \rangle = \sum_{x \in V} \mu(x) \psi_i(x) \psi_j(x) = \delta_{ij},$$

then we can write $f = \sum_{i=0}^{N-1} \alpha_i \psi_i$.

We obtain

$$-\Delta f = -\Delta \sum_{i=0}^{N-1} \alpha_i \psi_i = \sum_{i=0}^{N-1} \alpha_i \lambda_i \psi_i,$$

since

$$\sum_{i=0}^{N-1} \alpha_i \psi_i \cdot \sum_{i=0}^{N-1} \alpha_i \lambda_i \psi_i = \sum_{i=0}^{N-1} \alpha_i^2 \lambda_i \geq \lambda_1 f^2,$$

therefore

$$\int_V \Gamma(f)d\mu = -\int_V f\Delta f d\mu \geq \lambda_1 \int_V f^2 d\mu.$$

Let $C' \geq \frac{1}{\lambda_1}$, we may apply $\hat{f} = f - \int_V f d\mu$ to the above equality, we obtain the Poincaré inequality,

$$V(f) \leq C' \int_V \Gamma(f)d\mu.$$

Combining Lemma 2.1 and Lemma 2.2, we obtain the spectral inequality.

Step 3. Combining Theorem 1.1, step 1 and step 2, we obtain with the condition $CDE'(n,0)$, the tight log-Sobolev inequality $LS(C'')$ holds with

$$C'' = C''(n,K) = 2t_0 + C'(1 + \log M),$$

where $M = \frac{1}{\left(1-e^{-\frac{2K}{3}t_0}\right)^n}$ and $C' = 1536D_\mu D_\omega \pi \frac{n(5n+1)}{K}$. Minimizing the right-hand side of the above equality with respect to $t_0 > 0$, we have

$$t_0 = \frac{3}{2K}\log\left(1 + \frac{nKC'}{3}\right),$$

so we can get the result from simple computation.

3. Its applications

3.1. Exponential integrability

In this section we prove every integrable Lipschitz function is exponentially integrable if the Poincaré inequality and the tight log-Sobolev inequality holds respectively, moreover at the assumption of positive curvature on graphs. For a given Lipschitz function $f \in D(Q)$, we denote its Lipschitz norm by $\|f\|_{\text{Lip}} = \|\Gamma(f)\|_\infty^{\frac{1}{2}}$. A function is said to be 1-Lipschitz if $\|f\|_{\text{Lip}} \leq 1$.

Proposition 3.1. *Suppose that the graph satisfies the Poincaré inequality with a constant C'. Then, if f is 1-Lipschitz and $\int_V f d\mu < \infty$, we have*

$$\int_V e^{\lambda f} d\mu < \infty,$$

for any $\lambda < \sqrt{\frac{4}{C'}}$.

Proof. For any $n \in \mathbb{Z}^+$. let $\psi_n(t) = (-n) \vee t \wedge n$. It is easy to know $\psi_n(t)$ satisfies, for any $t_1, t_2 \in \mathbb{R}$

$$|\psi_n(t_1) - \psi_n(t_2)| \leq |t_1 - t_2|.$$

Considering $f_n(x) = \psi_n \circ f(x)$, which converges to $f(x)$ when $n \to \infty$, for every $x \in V$, we have $\Gamma(f_n, f_n) \leq \Gamma(f_n, f) \leq \Gamma(f, f) \leq 1$. Therefore, using Fatou's lemma, we may restrict ourselves to the case where f is bounded by replacing f to f_n.

Then we consider the function $g = e^{\frac{\lambda f}{2}}$, $\Gamma(f) \leq 1$. Since a important discrete estimation from Proposition 6.7 in [8], then

$$\int_V \Gamma(e^{\frac{\lambda f}{2}}) d\mu \leq \frac{\lambda^2}{2} \int_V e^{\lambda f} \Gamma(f) d\mu \leq \frac{\lambda^2}{2} \int_V e^{\lambda f} d\mu, \tag{3.1}$$

setting $\phi(\lambda) = \int_V e^{\lambda f} d\mu$, applied the the Poincaré inequality $P(C')$ to g, we obtain

$$\phi(\lambda)(1 - \frac{C'\lambda^2}{2}) \leq \phi^2(\frac{\lambda}{2}).$$

If $1 - \frac{C'\lambda^2}{2} > 0$, then we have

$$\phi(\lambda) \leq \phi^2(\frac{\lambda}{2})(1 - \frac{C'\lambda^2}{2})^{-1},$$

and it remains to iterate the procedure replacing λ by $\frac{\lambda}{2}$ to get the result.

Remark 3.2. We have, of course, a similar result to the general Lipschitz function f with $\Gamma(f) \leq c^2$ for changing f into $\frac{f}{c}$.

Proof of Theorem 1.3. As before, we may restrict ourselves to bounded 1-Lipschitz functions. Consider the function $\varphi(s) = \int_V e^{sf} d\mu$, observe that

$$\varphi'(s) = \int_V f e^{sf} d\mu,$$

while

$$E(e^{sf}) = s \int_V f e^{sf} d\mu - \varphi(s) \log \varphi(s) = s\varphi'(s) - \varphi(s) \log \varphi(s).$$

On the other hand, since $\Gamma(f) \leq 1$, and (3.1),

$$\int_V \Gamma(e^{\frac{sf}{2}}) d\mu \leq \frac{s^2}{2} \varphi(s).$$

Applying $LS(C)$ to $\varphi(s)$, then we have

$$s\varphi' - \varphi \log \varphi \leq Cs^2 \varphi,$$

integrating the above inequality. To this end, let $F(s) = \frac{1}{s} \log \varphi(s)$ (with $F(0) = \langle f \rangle$), so that

$$F'(s) = \frac{\frac{s\varphi'}{\varphi} - \log \varphi}{s^2} \leq C,$$

it follows,

$$F(s) \leq \int_V f d\mu + Cs,$$

which amounts to (1.1) immediately. Integrating (1.1) in the s with respect to the measure $e^{-\frac{s^2}{2\sigma^2}} ds$ on $\mathbb{R}$. Since

$$\int e^{tx - \frac{t^2}{2}} dt = e^{\frac{x^2}{2}},$$

by Fubini's Theorem, we have

$$\int_V e^{\frac{\sigma^2}{2} f^2} d\mu \leq \frac{1}{\sqrt{1 - 2C\sigma^2}} e^{\frac{\sigma^2 (\int_V f d\mu)^2}{2(1 - 2C\sigma^2)}} < \infty,$$

for every $\sigma^2 < \frac{1}{2C}$. Therefore the second claim holds and the proof ends.

3.2. Moment bounds on graphs

In this section, we obtain the moment bounds on graphs with positive curvature.

Proof of Theorem 1.4. As before, we first consider positive bounded functions. Consider the functional $\psi(p) = \|f\|_p^2$, $p > 2$, for a function $0 < f \in \ell^\infty(V, \mu)$ the derivative of $\psi(p)$ given by,

$$\psi'(p) = \frac{2}{p^2}\psi(p)^{1-\frac{p}{2}}E(f^p) \geq 0,$$

applying $LS(C)$ to $f^{\frac{p}{2}}$, combining the above equation, we obtain

$$\psi'(p) \leq \frac{4C}{p^2}\psi(p)^{1-\frac{p}{2}}\int_V \Gamma(f^{\frac{p}{2}})d\mu.$$

We need estimate the term $\int_V \Gamma(f^{\frac{p}{2}})d\mu$ for the symmetric property and Holdër inequality, since

$$\begin{aligned}\sum_{x\in V}\sum_{\substack{y\sim x\\ f(x)<f(y)}} \omega_{xy}(f(x)^{\frac{p}{2}} - f(y)^{\frac{p}{2}})^2 &= \\ &= \sum_{x\in V}\sum_{y\sim x} \omega_{xy}(f(x)^{\frac{p}{2}} - f(y)^{\frac{p}{2}})^2 \frac{1-\operatorname{sgn}(f(x)-f(y))}{2} = \\ &= \sum_{x\in V}\sum_{y\sim x} \omega_{yx}(f(y)^{\frac{p}{2}} - f(x)^{\frac{p}{2}})^2 \frac{1-\operatorname{sgn}(f(y)-f(x))}{2} = \\ &= \sum_{x\in V}\sum_{\substack{y\sim x\\ f(x)>f(y)}} \omega_{xy}(f(x)^{\frac{p}{2}} - f(y)^{\frac{p}{2}})^2,\end{aligned}$$

so we have

$$\begin{aligned}\int_V \Gamma(f^{\frac{p}{2}})d\mu &= \sum_{x\in V}\sum_{\substack{y\sim x\\ f(x)>f(y)}} \omega_{xy}(f(x)^{\frac{p}{2}} - f(y)^{\frac{p}{2}})^2 \leq \\ &\leq \sum_{x\in V}\sum_{\substack{y\sim x\\ f(x)>f(y)}} \omega_{xy}\left(\frac{p}{2}f(x)^{\frac{p}{2}-1}(f(x)-f(y))\right)^2 = \\ &= \frac{p^2}{4}\sum_{x\in V} f(x)^{p-2}\sum_{\substack{y\sim x\\ f(x)>f(y)}} \omega_{xy}(f(x)-f(y))^2 \leq \\ &\leq \frac{p^2}{2}\sum_{x\in V} f(x)^{p-2}\Gamma(f)(x) \leq \\ &\leq \frac{p^2}{2}\left(\sum_{x\in V}\mu(x)f^p(x)\right)^{1-\frac{2}{p}}\left(\sum_{x\in V}\mu(x)\Gamma(f)^{\frac{p}{2}}(x)\right)^{\frac{2}{p}} = \\ &= \frac{p^2}{2}\psi(p)^{\frac{p}{2}-1}\left(\int_V \Gamma(f)^{\frac{p}{2}}d\mu\right)^{\frac{2}{p}}.\end{aligned}$$

So substitute to the above inequality, we have

$$\psi'(p) \leq 2C \left(\int_V \Gamma(f)^{\frac{p}{2}} d\mu \right)^{\frac{2}{p}},$$

by integration with respect to p, and monotonicity, we can end the proof if f is bounded.

As before, in the general case then follows from consideration of f_n, since we know $\Gamma(f_n) \leq \Gamma(f)$, furthermore, $\|f_n\|_p \to \|f\|_p$ with $1 \leq p \leq \infty$ from $f_n \to f$ and $|f_n| \leq |f|$ for any $n \in \mathbb{Z}^+$. We complete what we desire.

REMARK

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ЛОГАРИФМИЧЕСКИЕ НЕРАВЕНСТВА СОБОЛЕВА НА ГРАФАХ ПОЛОЖИТЕЛЬНОЙ КРИВИЗНЫ

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Аннотация. В работе представлены некоторые важные неравенства на графах, такие как неравенство Пуанкаре и логарифмическое неравенство Соболева, а также плотное логарифмическое неравенство Соболева, полученные на основе глобальной оценки ядра уравнения теплопроводности при наложении только условия положительности кривизны $CDE'(n, K)$ с некоторым $K > 0$. В качестве следствий мы получили экспоненциальную интегрируемость интегрируемых липшицевых функций и границ момента на графах при том же предположении.

Ключевые слова: логарифмическое неравенство Соболева, лапласиан, $CDE'(n, K)$.