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CRYSTALLOGRAPHIC GROUPS FOR HÖRMANDER FIELDS

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Abstract. This is a preview paper on Crystallographic Groups of Hörmander Fields. We describe an emerging picture in analysis of extended groups. In particular, we introduce and provide examples of Crystallographic Groups associated to a Hörmander system of fields as well as discuss some related analysis.

Key words: extended Lie groups, noncommutative Dunkl-type operators, Markov semigroups, entropy & heat kernel bounds.

1. Introduction

In [78] we have introduced an infinite Coxeter group on two generators associated to the Heisenberg group $\mathbb{H}_1$ and studied related analysis. In this paper we present some possible outlook how that theory could be extended to include other noncompact Lie groups.

2. Crystallographic Groups

In this paper we will consider an algebraic and analytic structures associated to a given set of fields X_j, $j = 1, ..., N$, on a differentiable manifold G, given by a family of maps $\sigma_k : G \to G$, $k \in \mathbb{N}$, satisfying the following condition

$$X_j(f \circ \sigma_k) = \sum_{i,l} \alpha_{jk,il}(X_i f) \circ \sigma_l \tag{1}$$

(possibly only on certain sub-space of functions) with some $\alpha_{jk,il} \in \mathbb{R}$. In particular we introduce the following definition.

Definition 2.1. *A map* $\sigma : G \to G$ *satisfying*

$$X_j(f \circ \sigma) = -(X_j f) \circ \sigma, \qquad \sigma \circ \sigma = id \tag{2}$$

will be called a reflection.

3. Examples: Coxeter Groups of Systems of Fields

In this section we present explicit examples of maps indicated above for certain types of nilpotent Lie groups (as classified in section 4.1 in [15]).

3.1. B-Groups

Let $G \equiv \mathbb{R} \times \mathbb{R}^n$ with composition $\bullet$ of elements $w = (t, \mathbf{x}), w' = (t', \mathbf{x}')$ defined as follows

$$(t, \mathbf{x}) \bullet (t', \mathbf{x}') \equiv (t + t', \mathbf{x}' + e^{t\mathbf{B}}\mathbf{x})$$

Then for

$$\sigma(w) \equiv (-t, e^{-2t\mathbf{B}}\mathbf{x})$$

and

$$X \equiv \partial_t + (\mathbf{B}\mathbf{x}) \cdot \nabla,$$

where ∇ denotes the gradient with respect to $\mathbf{x}$, we have

$$X(f \circ \sigma) = -(Xf) \circ \sigma$$

For components ∂_j of ∇ the reflections are simply given by maps $x_j \to (-1)^{\delta_{ij}} x_j$, $j = 1,..,n$. the corresponding set (1) of maps includes also linear maps (with respect to $\mathbf{x}$).

3.2. K-Groups

Let $G \equiv \mathbb{R}^{n_0} \times \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r}$, with $n_0 \geq n_1 \geq .. \geq n_r$, $r \in \mathbb{N}$. This is similar to B-groups except that t is replaced by a n_0-vector $\mathbf{t}$ and the matrix in the definition of composition is lower triangular with the sub-diagonal part consisting of blocks $n_s \times n_{s-1}$ with rank equal to n_s for each $s = 1,..,r$. In this framework we consider the following fields (homogeneous of order one with respect to a natural gradation)

$$X_j \equiv \partial_{t_j} + (\mathbf{B}_j \mathbf{t}) \cdot \nabla$$

where ∇ denotes the gradient with respect to the variable of hidger degree of homogeneity (different than $\mathbf{t}$-variables).

Within this class we would like to indicate a special class of H-type groups defined with $r = 1$ and a single antisymmetric block B. In this case we have a nice family of reflection maps σ_j, $j = 1,..,n_0$, satisfying

$$X_j(f \circ \sigma_j) = -(X_j f) \circ \sigma_j$$

explicitly given by

$$\sigma_j(w) \equiv w \bullet (-2\{\delta_{ji}t_i\}, \mathbf{0}) = (\{(-1)^{\delta_{ji}}t_i\}, \mathbf{x}_k - 2\sum_i \mathbf{B}_{kji}t_j t_i)$$

with $j, i = 1,..,n_0$ and $k = 1,..,r$. These maps are idempotent, i.e. $\sigma_j \circ \sigma_j = id$, and (in general) they do not commute with each other. There are also some other reflection maps given as follows

$$\tilde{\sigma}_j(w) \equiv (\{(-1)^{\delta_{ji}}t_i\}, -\mathbf{x})$$

which do commute between themselves. It is interesting to notice that one can also have some other maps which intertwine the fields and are not of order two. To see this consider the following simplest example of such group.

3.3. $\mathbb{H}_1$ Group

With $w \bullet w' \equiv (x + x', y + y', z + z' + 2a(yx' - xy'))$, $a \neq 0$, and fields

$$X = \partial_x + 2a\pi_2\partial_z, \qquad Y = \partial_y - 2a\pi_1\partial_z$$

with $\pi_1(w) \equiv x$ and $\pi_2(w) \equiv y$, we have the following corresponding reflections

$$\begin{aligned} \sigma_X(w) &\equiv (-x, y, z - 4axy), \qquad & \tilde{\sigma}_X(w) &\equiv (-x, y, -z), \\ \sigma_Y(w) &\equiv (x, -y, z + 4axy), \qquad & \tilde{\sigma}_Y(w) &\equiv (x, -y, -z) \end{aligned}$$

and a common one

$$\hat{\sigma} \equiv (-x, -y, z).$$

Moreover, for

$$\sigma_+(w) \equiv (y, -x, z), \qquad \sigma_-(w) \equiv (-y, x, z)$$

we have

$$\sigma_- \circ \sigma_+(w) = id = \sigma_+ \circ \sigma_-, \qquad \sigma_-^{\circ 4} = id = \sigma_+^{\circ 4}, \qquad \sigma_-^{\circ 2} = \hat{\sigma} = \sigma_+^{\circ 2}.$$

We note also the following relations

$$\begin{aligned} \sigma_X \circ \tilde{\sigma}_X &= \tilde{\sigma}_X \circ \sigma_X, \qquad & \sigma_Y \circ \tilde{\sigma}_Y &= \tilde{\sigma}_Y \circ \sigma_Y, \\ \hat{\sigma} \circ \sigma_X &= \sigma_X \circ \hat{\sigma}, & \hat{\sigma} \circ \sigma_Y &= \sigma_Y \circ \hat{\sigma}, \\ \hat{\sigma} \circ \tilde{\sigma}_X &= \tilde{\sigma}_X \circ \hat{\sigma} = \tilde{\sigma}_Y, & \hat{\sigma} \circ \tilde{\sigma}_Y &= \tilde{\sigma}_Y \circ \hat{\sigma} = \tilde{\sigma}_X, \end{aligned}$$

and

$$\begin{aligned} \sigma_+ \circ \sigma_Y &= \sigma_X \circ \sigma_+, \qquad & \sigma_+ \circ \tilde{\sigma}_Y &= \tilde{\sigma}_X \circ \sigma_-, \\ \sigma_- \circ \sigma_Y &= \sigma_X \circ \sigma_-, & \sigma_- \circ \tilde{\sigma}_Y &= \tilde{\sigma}_X \circ \sigma_+, \\ \hat{\sigma} \circ \sigma_+ &= \sigma_+ \circ \hat{\sigma} = \sigma_-, & \hat{\sigma} \circ \sigma_- &= \sigma_- \circ \hat{\sigma} = \sigma_+. \end{aligned}$$

For σ_+, σ_-, we have

$$\begin{aligned} X(f \circ \sigma_+)(w) &= X(f(\pi_2, -\pi_1, \pi_3))(w) = \\ &= -(\partial_y f) \circ \sigma_+(w) + 2a\pi_2(w)(\partial_z f) \circ \sigma_+(w) = \\ &= (-(\partial_y f) + 2a\pi_1 \circ \sigma_-(\partial_z f)) \circ \sigma_+ = -(Yf) \circ \sigma_+(w) \end{aligned}$$

where we have used $\pi_2 \circ \sigma_- = \pi_1$. Similarly

$$\begin{aligned} Y(f \circ \sigma_-)(w) = Y(f(-\pi_2, \pi_1, \pi_3))(w) &= -(\partial_x f) \circ \sigma_-(w) - 2a\pi_1(\partial_z f) \circ \sigma_-(w) = \\ &= (-(\partial_x f) - 2a\pi_1 \circ \sigma_+(\partial_z f)) \circ \sigma_-(w) = -(Xf) \circ \sigma_-(w) \end{aligned}$$

where we have used $\pi_1 \circ \sigma_+ = \pi_2$. Finally for the reflection $\hat{\sigma}(w) \equiv (-x, -y, z)$, one has

$$\begin{aligned} X(f \circ \hat{\sigma}) = Xf(-\pi_1, -\pi_2, \pi_3) &= -(\partial_x f) \circ \hat{\sigma} + 2a\pi_2(\partial_z f) \circ \hat{\sigma} = \\ &= -(\partial_x f) \circ \hat{\sigma} - (2a\pi_2 \partial_z f) \circ \hat{\sigma} = -(Xf) \circ \hat{\sigma} \end{aligned}$$

and similarly

$$Y(f \circ \hat{\sigma}) = -(Yf) \circ \hat{\sigma}.$$

The Coxeter group generated by σ_X **and** σ_Y is of *infinite order* and all its elements can be describe as follows: We set

$$L_n \equiv (\sigma_X \sigma_Y)^n \quad L_0 \equiv id \quad L_{-n} \equiv (\sigma_Y \sigma_X)^n, \qquad n \in \mathbb{N}$$

and

$$R_{n+\frac{1}{2}} \equiv L_n \circ \sigma_X = \sigma_X \circ L_n, \qquad and \qquad R_{-n-\frac{1}{2}} \equiv L_{-n} \circ \sigma_Y = \sigma_Y \circ L_{-n}, \qquad n \in \mathbb{N}$$

With this notation we have

$$L_n \circ L_m = L_{n+m}, \qquad and \qquad R_n \circ L_m = R_{n+m}.$$

The subgroup $\{L_n\}_{n\in\mathbb{Z}}$ is abelian and moreover any (group) commutator in the group generated by σ_X, σ_Y belongs to $\{L_n\}_{n\in\mathbb{Z}}$. The group is furnished with the following intertwining structure

$$\sigma_\pm \circ R_{n+\frac{1}{2}} = R_{-n-\frac{1}{2}} \circ \sigma_\pm \quad and \quad \sigma_\pm \circ L_n = I_{-n} \circ \sigma_\pm$$

Finally we mention that the fields X and Y are preserved by the following families of translations

$$\tau_X(w) \equiv (x + \mathrm{a}, y, z + \mathrm{c}), \qquad \tau_Y(w) \equiv (x, y + \mathrm{b}, z + \mathrm{c}),$$

respectively, with $\mathrm{a}, \mathrm{b}, \mathrm{c} \in \mathbb{R}$, as well as the left action of the group.
Remark. In complex representation of Heisenberg group where $w \equiv (v, t)$ one has the following natural reflections $\sigma_{*,\pm}(w) \equiv (v^*, \pm t)$, $\hat{\sigma}(w) \equiv (-v, t) = \sigma_\pm^{\circ 2}(w)$ where $\sigma_\pm(w) = (\mp iv, t)$. Then $\tilde{\sigma}_X(w) = \hat{\sigma} \circ \tilde{\sigma}_Y(w)$ with $\tilde{\sigma}_Y(w) = \sigma_{*,-}(w)$. Finally $\sigma_Y(w) = (v^*, t + 2a\Re(-iv^2))$ and $\sigma_X = \sigma_+ \circ \sigma_Y \circ \sigma_-$.

3.4. Examples of Higher Order

(i) Let $G = \mathbb{R}^4$ and

$$w \bullet w' = (x_1 + x_1', x_2 + x_2', x_3 + x_3' + x_1' x_2, x_4 + x_4' + x_1' x_2^2 + 2x_2 x_3')$$

with dilation $\delta_\lambda(w) = (\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda^3 x_4)$. Consider the following system of fields

$$X_1 \equiv \partial_1 + x_2 \partial_3 + x_2^2 \partial_4, \qquad X_2 \equiv \partial_2.$$

See e.g. [15] for such the structure. Define

$$\sigma_1(w) \equiv w \bullet (-2x_1, 0, 0, 0) = (-x_1, x_2, x_3 - 2x_1 x_2, x_4 - 2x_1 x_2^2).$$

Then we have $\sigma_1 \circ \sigma_1 = id$ and

$$\begin{aligned} X_1(f \circ \sigma_1) &= X_1(f(-\pi_1, \pi_2, \pi_3 - 2\pi_1\pi_2, \pi_4 - 2\pi_1\pi_2^2)) = \\ &= -(\partial_1 f) \circ \sigma_1 - 2\pi_2(\partial_3 f) \circ \sigma_1 - 2\pi_2^2(\partial_4 f) \circ \sigma_1 + \pi_2(\partial_3 f) \circ \sigma_1 + \pi_2^2(\partial_4 f) \circ \sigma_1 = \\ &= -(X_1 f) \circ \sigma_1. \end{aligned}$$

Also for

$$\sigma_1'(w) \equiv (-x_1, x_2, -x_3, -x_4), \qquad \hat{\sigma} \equiv (-x_1, -x_2, x_3, -x_4)$$

one gets
$$X_1(f \circ \sigma_1') = -(X_1 f) \circ \sigma_1', \quad and \quad X_1(f \circ \hat{\sigma}) = -(X_1 f) \circ \hat{\sigma}.$$
In all cases we have $id = \sigma_1 \circ \sigma_1 = \sigma_1' \circ \sigma_1' = \hat{\sigma}_1 \circ \hat{\sigma}_1$. For the second field the relation
$$X_2(f \circ \sigma_2) = -(X_2 f) \circ \sigma_2$$
is satisfied with any map of the form
$$\sigma_2(w) \equiv (\alpha x_1, -x_2 + \beta x_1, \gamma x_3 + \delta x_1^2, \varepsilon x_4 + \zeta x_1 x_3)$$
with $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$. Such the map satisfies condition $\sigma_2 \circ \sigma_2 = id$ if additionally one requests $(\alpha, \delta, \gamma, \varepsilon) = (-1, 0, 1, 1), (-1, 0, -1, -1)$ or $(\alpha, \beta, \delta, \gamma, \varepsilon) = (1, 0, 0, -1, 1)$ and $(\alpha, \beta, \delta, \gamma, \varepsilon, \zeta) = (1, 0, 0, 1, -1, 0)$. For the last choice we obtain $\hat{\sigma}$ which also reflects the first field. This is a situation which is different from the $\mathbb{H}_1$ group.

(ii) Let $G = \mathbb{R}^4$ and
$$w \bullet w' = (x_1 + x_1', x_2 + x_2', x_3 + x_3' + x_1' x_2, x_4 + x_4' + \frac{1}{2} x_2 (x_1')^2 + x_3 x_1')$$
with dilation $\delta_\lambda(w) = (\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda^3 x_4)$. Consider the following system of fields
$$X_1 \equiv \partial_1 + x_2 \partial_3 + x_3 \partial_4, \qquad X_2 \equiv \partial_2,$$
see e.g. [15]. Define
$$\sigma_1(w) \equiv w \bullet (-2x_1, 0, 0, 0) = (-x_1, x_2, x_3 - 2x_1 x_2, x_4 + 2x_1^2 x_2 - 2x_1 x_3).$$
Then we have
$$\begin{aligned} X_1(f \circ \sigma_1) = X_1(f(-\pi_1, \pi_2, \pi_3 - 2\pi_1\pi_2, \pi_4 + 2\pi_1^2\pi_2 - 2\pi_1\pi_3)) = \\ = -(\partial_1 f) \circ \sigma_1 - \pi_2 (\partial_3 f) \circ \sigma_1 + (2\pi_1\pi_2 - \pi_3)(\partial_4 f) \circ \sigma_1 = \\ = -(X_1 f) \circ \sigma_1. \end{aligned}$$
For
$$\sigma_1'(w) \equiv (-x_1, x_2, -x_3, x_4) \quad or \quad \hat{\sigma} \equiv (-x_1, -x_2, x_3, -x_4)$$
we get also reflection of the field X_1. In all cases we have $id = \sigma_1 \circ \sigma_1 = \sigma_1' \circ \sigma_1' = \hat{\sigma} \circ \hat{\sigma}$. For the second field we have
$$X_2(f \circ \sigma_2) = -(X_2 f) \circ \sigma_2$$
with any map of the form
$$\sigma_2(w) \equiv (\alpha x_1, -x_2 + \beta x_1, \gamma x_3 + \delta x_1^2, \varepsilon x_4 + \zeta x_1 x_3)$$
with $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$. Such the map satisfies condition $\sigma_2 \circ \sigma_2 = id$, if additionally one requests $(\alpha, \delta, \gamma, \varepsilon) = (-1, 0, 1, 1), (-1, 0, -1, -1)$ or $(\alpha, \beta, \delta, \gamma, \varepsilon) = (1, 0, 0, -1, 1)$. We remark that in the present case also $\hat{\sigma}$ reflects X_2.

4. Problem of Effective Description of Coxeter Groups

The first emerging general issue is as follows.

(P1.i) Given a basis X_j, $j = 1, .., N$, of free nilpotent Lie algebra, as described in [48], describe the Coxeter groups CX_j of a given field X_j.

(P1.ii) Describe the crystallographic group corresponding to the basis X_j, $j = 1, .., N$.

In the second case we ask for more than just the description of the group $\bigvee_j CX_j$ generated by all CX_j, as it could also include intertwiners as well as possibly some other elements. One can illustrate that a full variety of structures can occur by considerig the following class of examples of n fields with linear coefficients in $\mathbb{R}^{n+1}$

$$X_j \equiv \frac{\partial}{\partial x_j} + \sum_{i \neq j} B_{ji} x_i \frac{\partial}{\partial z},$$

where B_{ji} is a nonsymmetric matrix, with associated reflections given by

$$\sigma_j(w) \equiv (\{(-1)^{\delta_{ij}} x_i\}_{i=1,..,n}, z - 2\sum_{i \neq j} B_{ij} x_j x_i).$$

The commutativity structure can be obtained by constructing a graph with n vertices in which vertices v_i and v_j are connected by an edge e_{ij} iff $B_{ij} \neq 0$, and in this case σ_i and σ_j do not commute. $\sigma_j, j = 1, .., n$, generate a Coxeter group of infinite order. Additionally in the present case CX_j contains reflections

$$\tilde{\sigma}_j(w) \equiv (\{(-1)^{\delta_{ij}} x_i\}_{i=1,..,n}, -z)$$

and

$$\hat{\sigma}_j(w) \equiv (\{(-1)^{\delta_{ij}} (-1)^{\delta_{e_{ij} \neq \emptyset}} x_i\}_{i=1,..,n}, z)$$

and possibly some others (including also number of intertwiners). Moreover the field X_j is invariant with respect to Coxeter group acting on variables which are not involved in definition on X_j as well as with respect to traslations $\tau_j(w) \equiv (\{x_i + \alpha_j \delta_{ij}\}_{i=1,..,n}, z + \gamma_j)$, with $\alpha_j, \gamma_j \in \mathbb{R}$, respectively.

Generally we may have a finite Coxeter subgroup and additional infinite component in a given Coxeter group. We remark that in some examples we had a subgroup containing reflections of many fields. One may expect that these data could help to classify the corresponding noncompact Lie groups.

5. Inverse Problem for Coxeter Groups

Suppose one is given generators σ_j of a Coxeter group on $\mathbb{R}^N$, then one can ask what are the fields X which satisfy the reflection relation

$$(Xf \circ \sigma_j) = -(Xf) \circ \sigma_j. \tag{**}$$

For example one knows (see e.g. [70]) that the Painlevé II

$$\ddot{u} = 2u^3 + tu + b - \frac{1}{2},$$

represented as a Hamiltonian system with canonical variables

$$q \equiv u, \qquad p \equiv \dot{u} + u^2 + \frac{t}{2}$$

as follows

$$\begin{array}{l} \dot{q} = \frac{\partial H}{\partial p} = p - q^2 - \frac{t}{2} \\ \dot{p} = -\frac{\partial H}{\partial q} = 2pq + b \end{array}$$

admits a Coxeter group of Bäcklund transformations generated by the following map

$$\begin{aligned} \sigma_1(q,p,b) &\equiv (q + \frac{b}{p}, p, -b) \\ \sigma_2(q,p,b) &\equiv (-q, -p + 2q^2 + t, 1 - b). \end{aligned}$$

It is the infinite dihedral group encountered before for the Heisenberg group $\mathbb{H}_1$. One can solve the reflection equation with respect to the fields getting

$$\begin{aligned} X_1(f \circ \sigma_1) &= -(X_1 f) \circ \sigma_1, \qquad X_1 = \frac{\partial}{\partial b} + (-\frac{1}{2p} + \alpha_1)\frac{\partial}{\partial q} + \beta_1 \frac{\partial}{\partial p}, \\ X_2(f \circ \sigma_2) &= -(X_2 f) \circ \sigma_2, \qquad X_2 = \frac{\partial}{\partial q} + (-2q)\frac{\partial}{\partial p} + \gamma_2 \frac{\partial}{\partial b} \end{aligned}$$

with the functions $\alpha_1 \circ \sigma_1 = -\alpha_1$, $\beta_1 \circ \sigma_1 = \beta_1$ and $\gamma_2 \circ \sigma_2 = \gamma_2$. (It would be interesting to repeat that for other Pailevé problems using identification of [40; 61].)

Homogeneous Lie groups on $\mathbf{R}^n$

Consider now the case when the group action is given by

$$(w \bullet w')_k \equiv x_k + x'_k + Q_k(w, w')$$

with $Q_k(w, w') \equiv Q_k(x_j, x'_j : j = 1, .., k-1)$, satisfying

$$Q_k(w, w')_{|x'_j = 0 : j = 1,..,k-1} = 0.$$

(According to Theorem 1.3.15 in [15] such properties would satisfy a composition law for any homogeneous Lie group on $\mathbb{R}^n$.) Then, each map

$$\sigma_i(w) \equiv w \bullet \mathrm{r}^{(i)}(w), \qquad \text{with } (\mathrm{r}^{(i)}(w))_j \equiv -2\delta_{i,j} x_i, \tag{3}$$

is a reflection, i.e. $\sigma_i \circ \sigma_i = id$, if the inverse to $\mathrm{r}^{(i)}(w)$ is given by $-\mathrm{r}^{(i)}(w)$. This is because $(\mathrm{r}^{(i)}(w \bullet \mathrm{r}^{(i)}(w)))_j = -2\delta_{i,j}(w \bullet \mathrm{r}^{(i)}(w))_i = 2\delta_{i,j} x_i$ and this defines the inverse element to $\mathrm{r}^{(i)}(w)$ in our group. (In fact, by Proposition 2.2.22 in [15], for stratified Lie groups in appropriate representation this is always the case.)

In this case one can study the corresponding reflection problem $(**)$ for fields

$$X_i f = \lim_{\varepsilon \to 0} \frac{f(w \circ \varepsilon_i) - f(w)}{\varepsilon} = \frac{d}{d\varepsilon} f(w \circ \varepsilon_i)_{\varepsilon = 0}.$$

We have the following result.

Theorem 5.1. *For σ_i given by (3) the fields X_i satisfy the reflection problem $(**)$ for, respectively, provided that*

$$\left[\frac{d}{d\varepsilon}Q_k\left((\sigma_i(w)\circ\varepsilon_i)_{l\leq k-1},(-2\delta_{ij}(\sigma_i(w)_i+\varepsilon))_{j\leq k-1}\right)\right]_{|\varepsilon=0} = \\ = -\left[\frac{d}{d\varepsilon}Q_k((\sigma_i(w))_{l\leq k-1},(\delta_{ij}\varepsilon)_{j\leq k-1})\right]_{|\varepsilon=0}.$$

Proof. Using the definition of the fields X_i, we have

$$X_i(f\circ\sigma_i)=\lim_{\varepsilon\to 0}\frac{f\circ\sigma_i(w\circ\varepsilon_i)-f\circ\sigma_i(w)}{\varepsilon}$$

with

$$\begin{aligned}(\sigma_i(w\circ\varepsilon_i))_k &= (w\circ\varepsilon_i\circ \mathrm{r}^{(i)}(w\circ\varepsilon_i))_k\\ &= (w\circ\varepsilon_i)_k-2\delta_{ik}(w\circ\varepsilon_i)_i+Q_k(w\circ\varepsilon_i,\mathrm{r}^{(i)}(w\circ\varepsilon_i))\\ &= (-1)^{\delta_{ik}}(w\circ\varepsilon_i)_k+Q_k(w\circ\varepsilon_i,\mathrm{r}^{(i)}(w\circ\varepsilon_i)).\end{aligned}$$

Using $(\mathrm{r}^{(i)}(w\circ\varepsilon_i))_k=-2\delta_{ik}(w_i+\varepsilon)$ and our assumption about Q_k's, we have

$$(\sigma_i(w\circ\varepsilon_i))_k=\begin{cases} w_k, & for\ k<i\\ -w_i-\varepsilon, & for\ k=i\\ w_k+Q_k((w\circ\varepsilon_i)_{l\leq k-1},(-2\delta_{ij}(w_i+\varepsilon))_{j\leq k-1}), & for\ k>i\end{cases}$$

Hence

$$X_i(f\circ\sigma_i)(w)=-(\partial_i f)\circ\sigma_i(w)\\ +\sum_{k>i}\left[\frac{d}{d\varepsilon}Q_k((w\circ\varepsilon_i)_{l\leq k-1},(-2\delta_{ij}(w_i+\varepsilon)\Big)_{j\leq k-1})\right]_{|\varepsilon=0}(\partial_k f)\circ\sigma_i(w)$$

while we have

$$(X_i f)\circ\sigma_i(w)=(\partial_i f)\circ\sigma_i(w)+\\ +\sum_{k>i}\left[\frac{d}{d\varepsilon}Q_k((\sigma_i(w))_{l\leq k-1},(\delta_{ij}\varepsilon)_{j\leq k-1})\right]_{|\varepsilon=0}(\partial_k f)\circ\sigma_i(w).$$

Thus to get (**) we need

$$\left[\frac{d}{d\varepsilon}Q_k\left((\sigma_i(w)\circ\varepsilon_i)_{l\leq k-1},(-2\delta_{ij}(\sigma_i(w)_i+\varepsilon))_{j\leq k-1}\right)\right]_{|\varepsilon=0} = \\ = -\left[\frac{d}{d\varepsilon}Q_k((\sigma_i(w))_{l\leq k-1},(\delta_{ij}\varepsilon)_{j\leq k-1})\right]_{|\varepsilon=0}.$$

We remark that in non-homogeneous case one can allow for maps having a component

$$x_j\mapsto -x_j+q_j(x_k,k<j)$$

with even polynomial q_j if the components $x_k, k < j$, just change a sign, as we have seen it in case of Bäcklund transformation.

Remark. In complex representation of Heisenberg group where $w \equiv (v, t)$ one has also the following natural reflection $\bar{\sigma}_Y(w) = (v^*, t - 2a\Re(v^2))$ and $\bar{\sigma}_X(w) = (-x, y, t + 2a\Re(v^2))$. Then the fields

$$\bar{Y} \equiv \partial_y - 2ay\partial_z, \qquad \bar{X} \equiv \partial_x - 2ax\partial_z$$

would satisfy

$$\bar{Y}(f \circ \bar{\sigma}_Y) = -(\bar{Y}f) \circ \bar{\sigma}_Y, \qquad \bar{X}(f \circ \bar{\sigma}_X) = -(\bar{X}f) \circ \bar{\sigma}_X.$$

It would be interesting to find out for what Coxeter groups one can choose a system of fields such that (**) holds and for which Hörmander condition is satisfied.

6. Representations and Co- & In-variant Functions

In [78] where we have studied $\mathbb{H}_1$ and introduced

$$\sigma_X(w) \equiv (-x, y, z - 4axy), \qquad \sigma_Y(w) \equiv (x, -y, z + 4axy)$$

we have provided there a simple representation of the corresponding Coxeter group on the linear span of the following generalised linear functions

$$x, y, \eta \equiv x + \frac{z}{2ay}, \zeta \equiv y - \frac{z}{2ax}$$

as follows

$$\begin{array}{llll} x \circ \sigma_X = -x & y \circ \sigma_X = y & \eta \circ \sigma_X = \eta - 4x & \zeta \circ \sigma_X = -\zeta \\ x \circ \sigma_Y = x & y \circ \sigma_Y = -y & \eta \circ \sigma_Y = -\eta & \zeta \circ \sigma_Y = \zeta - 4y \end{array}$$

To this we add relations coming from additional σ's described above as follows

$$\begin{array}{lll} \tilde{\sigma}_X(w) \equiv (-x, y, -z), & \tilde{\sigma}_Y(w) \equiv (x, -y, -z), & \hat{\sigma}(w) \equiv (-x, -y, z) \\ & \sigma_+(w) \equiv (y, -x, z), & \sigma_-(w) \equiv (-y, x, z) \end{array}$$

for which we get

$$\begin{array}{ll} \eta \circ \hat{\sigma} = -\eta, & \zeta \circ \hat{\sigma} = -\zeta \\ \eta \circ \tilde{\sigma}_X = -\eta, & \zeta \circ \tilde{\sigma}_X = \zeta \\ \eta \circ \tilde{\sigma}_Y = \eta, & \zeta \circ \tilde{\sigma}_Y = -\zeta \end{array}$$

and the following intertwining relation

$$\eta \circ \sigma_+ = \zeta.$$

The Coxeter group generated by $\tilde{\sigma}_X, \tilde{\sigma}_Y, \sigma_\pm$ preserves the Kaplan norm, while a group including also generators σ_X, σ_Y would preserve the set of functions spanned by the following

$$\phi_q(w) \equiv \sum_q a_q \sum_{n \in \mathbb{Z}} \xi_q^n \, e^{i\lambda(z - n\frac{4a}{q}xy)} \phi(z - n\frac{4a}{q}xy))$$

defined with $q \in \mathbb{N}$ and $(\xi_q)^q = 1$, $\xi_q \neq 1$, $\lambda \in \mathbb{R}$ and a function ϕ.

Note that

$$\phi_q \circ \sigma_X(w) = \sum_{n\in\mathbb{Z}} \xi_q^n \, e^{i\lambda(z+n\frac{4a}{q}xy)} \phi(z + n\frac{4a}{q}xy))$$

and

$$\phi_q \circ \sigma_Y(w) = \sum_{n\in\mathbb{Z}} \xi_q^n \, e^{i\lambda(z+n\frac{4a}{q}xy)} \phi(z + n\frac{4a}{q}xy)).$$

For higher order examples considered in section 3, besides homogeneous norms, one can find also for example the following homogeneous polynomials invariant with respect to the basic finite Coxeter subgroup (not including σ_1)

(i) $x_1x_2x_3 + x_1x_4 + x_3^2 + x_2^4 + x_1^4$

(ii) $x_1x_2x_3 + x_2x_4 + x_3^2 + x_2^4 + x_1^4$

which are very similar. This brings us to the following problems.

Problems 2.

(**P2.i**) Describe representations of the groups $\bigvee_j CX_j$ on spaces of functions.

(**P2.ii**) Describe the invariant homogeneous polynomials with respect to subgroups as well as fully invariant functions.

7. Linear Operators

Given $\sigma \in CX_j$, $\sigma^{\circ 2} = id$, we can introduce the following linear operators.

DeMazur Operators

$$A_{j,\sigma} \equiv \frac{\kappa_\sigma}{\eta_\sigma}(I - I_\sigma)$$

with $I_\sigma f \equiv f \circ \sigma$ and a differentaible function η_σ satisfying

$$\eta_\sigma \circ \sigma = -\eta_\sigma \qquad and \qquad X_j\eta_\sigma = 1,$$

and a constant $\kappa_\sigma \neq 0$. First we can introduce it on smooth functions vanishing on an open set containing the set $\mathfrak{S}_\sigma \equiv \{\eta_\sigma = 0\}$, and later extend by continuity. One can see that $A_{j,\sigma}$ vanishes on σ-symmetric functions (i.e. functions satisfying $I_\sigma f \equiv f \circ \sigma = f$), and we have the following:

Boundary Operator Property

$$A_{j,\sigma}^2 = 0.$$

Using DeMazur operator we define the following 1st order operators:

Generalised Fields

$$T_{j,\sigma} \equiv X_j + A_{j,\sigma}.$$

Some care is necessary here as we are adding two unbounded operators. In case of free nilpotent Lie group when the reflection has a simple implementation (with group left multiplication by $(-2\delta_{ij}x_j)$), we note that

$$A_{j,\sigma}f(w) = 2\kappa_j \int_0^1 ds (X_jf)(\gamma_s^w)$$

with a path γ_s^w connecting w and $\sigma(w)$. Since by choice of our σ we have the following.

Anticommutation Relations

$$\{X_j, I_\sigma\} \equiv X_j I_\sigma + I_\sigma X_j = 0, \qquad \{A_{j,\sigma}, I_\sigma\} = 0,$$

so also

$$\{T_{j,\sigma}, I_\sigma\} = 0.$$

More generally one could introduce

$$T_j \equiv X_j + \sum_{\sigma \in CX_j} \beta_\sigma A_{j,\sigma}$$

with $\beta_\sigma \in \mathbb{R}$, $\sum_{\sigma \in CX_j} \beta_\sigma = 1$.
With these extended fields, assuming later on $\beta_\sigma, \kappa_\sigma \geq 0$, we introduce the following second order operator

$$\mathcal{L} \equiv \sum_{j,\sigma} T_{j,\sigma}^2.$$

In this case one could check, similarly as in [78], (see also section 10), that such operator satisfy the following.

Proposition 7.1 (Minimum Principle).

$$f(w_0) = \min f \Longrightarrow \mathcal{L} f(w_0) \geq 0$$

Proof. Since $A_\sigma^2 = 0$, we have

$$\mathcal{L} f = \sum_j \left(X_j^2 + \{X_j, A_{\sigma_j}\} \right) f.$$

If a minimum point $x_0 \notin \cup_\sigma \mathfrak{S}_\sigma$ we have

$$X_j A_{\sigma_j} f(x_0) = \frac{\kappa_{\sigma_j}}{\eta_{\sigma_j}(x_0)} \left(X_j f(x_0) - X_j (f \circ \sigma_j)(x_0) \right) - \frac{\kappa_{\sigma_j} X_j \eta_{\sigma_j}(x_0)}{\eta_{\sigma_j}^2(x_0)} A_{\sigma_j} f(x_0)$$

and

$$A_{\sigma_j} X_j f = \frac{\kappa_{\sigma_j}}{\eta_{\sigma_j}(x_0)} \left(X_j f(x_0) - (X_j f) \circ \sigma_j(x_0) \right).$$

Hence, using reflection property, we get

$$\{X_j, A_{\sigma_j} f\}(x_0) = \frac{2\kappa_{\sigma_j}}{\eta_{\sigma_j}(x_0)} X_j f(x_0) - \frac{\kappa_{\sigma_j} X_j \eta_{\sigma_j}(x_0)}{\eta_{\sigma_j}^2(x_0)} A_{\sigma_j} f(x_0).$$

The first term on the right hand side is equal to zero at the minimum point. If $X_j \eta_{\sigma_j} \geq 0$ as we assume, we have

$$-\frac{\kappa_{\sigma_j} X_j \eta_{\sigma_j}(x_0)}{\eta_{\sigma_j}^2(x_0)} A_{\sigma_j} f(x_0) = \frac{\kappa_{\sigma_j} X_j \eta_{\sigma_j}(x_0)}{\eta_{\sigma_j}^2(x_0)} \left(f(\sigma_j x_0) - f(x_0) \right) \geq 0$$

because at the minimum point $f(\sigma_j x_0) \geq f(x_0)$. In case the minimum point $x_0 \in \cup_\sigma \mathfrak{S}_\sigma$, we need to use limiting procedure necessary to extend definition of A_σ to reflection ivariant points. As we pointed out above (in case of nilpotent Lie groups) when $X_j \eta_{\sigma_j} = 1$, one gets

$$\{X_j, A_{\sigma_j} f\}(x_0) = 2\kappa_j X_j^2 f(x_0) \geq 0$$

as required.

We note that, as in the case of Heisenberg group, the Coxeter group generated by the reflections of the basic fields can be infinite and may contain infinitely many reflections of the space subordinated to some other fields. For example in the Heisenberg case one generates reflections of the form $(\sigma_X \sigma_Y)^n \sigma_X$ and $(\sigma_Y \sigma_X)^n \sigma_Y$, $n \in \mathbb{N}$, which would satisfy the reflection relation with the fields of the form $X_k \equiv \partial_x + 2kay\partial_z$ or $Y_k \equiv \partial_y - 2kax\partial_z$, for suitable $k \in \mathbb{Z}$. In this case we get discrete infinite family of representations of the Heisenberg-Lie algebra and in this case one may also consider the following Markov generator

$$\mathcal{L} \equiv \sum_{j,k,\sigma} \varepsilon_k T_{j,k,\sigma}^2$$

with $\varepsilon_k \in (0, \infty)$ satisfying $\sum_k \varepsilon_k < \infty$.

Proposition 7.2. *Assume the fields X_i satisfy integration by parts formula with a measure ν invariant with respect all σ_i. Let ρ be a density function with respect to ν. If*

$$X_i \log \rho = 2\kappa_i \eta_i^{-1} \tag{4}$$

for all i (almost everywhere), then for all $T_i \equiv X_i + A_i$ the following generalised integration by parts formula holds

$$\int T_i(g) f \rho d\nu = -\int g T_i(f) \rho d\nu. \tag{5}$$

Proof. Since for a given σ_i invariant measure ν, we have

$$\int A_i(g) f d\nu = -\int g A_i(f) d\nu + \int g \kappa_i \eta_i^{-1} f d\nu.$$

Using this together with the integration parts formula for the field X_i with the measure ν, we notice that to have (5), one would need to satisfy the following condition

$$X_i \log \rho = 2\kappa_i \eta_i^{-1} \tag{6}$$

for all i (almost everywhere).

If the fields X_i's generate nilpotent Lie algebra, (6) provides necessary conditions on η_i's for solvability, (as applying other fields and forming linear combination and repeating this many times one can generate commutators of sufficiently high order which have to vanish by nilpotency condition). Also applying X_i again to (6) and summing over i's one obtains a relation

$$(\sum_i X_i^2) \log \rho = \sum_i 2\kappa_i X_i \eta_i^{-1}. \tag{7}$$

In particular if $X_i\eta_i = 1$ we get, with $\mathbb{X}\cdot\mathbb{X} \equiv (\sum_i X_i^2)$, the following

$$\mathbb{X}\cdot\mathbb{X}\log\rho = -\sum_i 2\kappa_i\eta_i^{-2}. \tag{8}$$

In case when X_i are generators of free nilpotent Lie group, we have $X_i = \partial_i + \sum_{j>i}\alpha_j\partial_j$ with α_j dependent on the proceeding coordinates, with $\eta_j \equiv x_j$ one can choose

$$\rho = \prod_i x_i^{2\kappa_i}. \tag{9}$$

A possibility of other choices of η's is discussed later in section 8.2.

Canonical Markovian and Dirichlet Forms

When the generalised integration by parts formula holds with measure $d\mu \equiv \rho d\nu$, we have

$$-\int g\mathcal{L}f d\mu = \int \mathbb{T}g\cdot\mathbb{T}f d\mu \equiv \sum_i \int (T_i g)T_i f d\mu.$$

On the other hand the canonical Markovian form of $\mathcal{L}$ is as follows

$$\Gamma(f) \equiv \frac{1}{2}(\mathcal{L}f^2 - 2f\mathcal{L}f) = |\mathbb{X}f|^2 + \sum_i \frac{X_i\eta_i}{2\kappa_i}(A_i f)^2.$$

If $X_i\eta_i \geq 0$, then this form is clearly nonnegative. (In particular this is satisfied for generators of free nilpotent Lie groups with η's given by corresponding coordinate functions.) In this situation the right hand side of (7) is nonnegative.

If the generalised integration by parts formula holds, we have

$$\int \Gamma(f)d\mu = \int \mathbb{T}g\cdot\mathbb{T}f d\mu \geq \int |\mathbb{X}f|^2 d\mu \tag{10}$$

with the left hand side satisfying evidently positivity and contraction property of a Dirichlet form on a suitable dense space of functions. Thus, after closure, it defines a Markov generator in $L_2(\mu)$.

Next we remark that for density ρ and a differentiable function f, we have by Leibnitz rule $\mathbb{X}(f\rho^{1/2}) = (\mathbb{X}f)\rho^{1/2} + \frac{1}{2}f\rho^{1/2}(\mathbb{X}\log\rho)$ and hence for $d\mu = \rho d\nu$, we get

$$\int |\mathbb{X}f|^2 d\mu \geq -\int \mathbb{X}(f\rho^{1/2})\cdot f\rho^{1/2}(\mathbb{X}\log\rho)d\nu + \int f^2\frac{1}{4}|\mathbb{X}\log\rho|^2 d\mu. \tag{11}$$

From this, by integration by parts in the first term on the right hand side, we arrive at the following.

Proposition 7.3 (Hardy type inequality I).

$$\int |\mathbb{X}f|^2 d\mu \geq \int f^2\left(\frac{1}{4}|\mathbb{X}\log\rho|^2 + \frac{1}{2}\mathbb{X}\cdot\mathbb{X}\log\rho\right)d\mu. \tag{12}$$

Using the condition (6)–(7) we thus obtain.

Proposition 7.4 (Hardy type inequality II). *For generators of nilpotent Lie groups with corresponding* ρ *given by (9) with* $\kappa_i > 1$

$$\int |\mathbb{T}f|^2 d\mu \geq \int f^2 \sum_i (\kappa_i^2 \eta_i^{-2} + \kappa_i X_i \eta_i^{-1}) d\mu. \tag{13}$$

In particular if $X_i \eta_i = 1$ *one gets*

$$\int |\mathbb{T}f|^2 d\mu \geq \int f^2 \sum_i \kappa_i(\kappa_i - 1)\eta_i^{-2} d\mu. \tag{14}$$

Since the set of Hörmander fields satisfy Sobolev inequality, one also has the following log-Sobolev type inequality satisfied

$$\int f^2 \log \frac{f^2}{\int f^2 d\nu} d\nu \leq \varepsilon \int |\mathbb{X}f|^2 d\nu + C(\varepsilon) \int f^2 d\nu$$

for any $\varepsilon \in (0, \infty)$ with $C(\varepsilon) \sim const \log \frac{1}{\varepsilon}$ for small ε. In particular substituting in this inequality a function $f\rho^{\frac{1}{2}}$, we get

$$\int f^2 \log \frac{f^2}{\int f^2 d\mu} d\mu + \int f^2 \log \rho d\mu \leq \varepsilon \int |\mathbb{X}f|^2 d\mu + \varepsilon \int f\mathbb{X}f \cdot \mathbb{X} \log \rho \; d\mu + \\ + \frac{\varepsilon}{4} \int f^2 \cdot |\mathbb{X} \log \rho|^2 \; d\mu + C(\varepsilon) \int f^2 d\mu.$$

Hence

$$\int f^2 \log \frac{f^2}{\int f^2 d\mu} d\mu + \int f^2 \log \rho d\mu \leq \varepsilon \int |\mathbb{X}f|^2 d\mu - \\ -\varepsilon \int f^2 \left(\frac{1}{4} |\mathbb{X} \log \rho|^2 + \frac{1}{2} \mathbb{X} \cdot \mathbb{X} \log \rho \right) \; d\mu + \\ + C(\varepsilon) \int f^2 d\mu$$

Using Hardy inequality together with quadratic form bound (10) one can dress this inequality up and obtain, (as in [78]), the following result.

Theorem 7.5 (Log-Sobolev Coercive Bound). *Suppose Hardy Inequality (12) holds and suppose for any* $\varepsilon \in (0, 1)$ *we have*

$$\log_+ \rho^{-1} \leq C\varepsilon \left(\frac{1}{4} |\mathbb{X} \log \rho|^2 + \frac{1}{2} \mathbb{X} \cdot \mathbb{X} \log \rho \right) \; + DC(\varepsilon)$$

with some constants $C, D \in [0, \infty)$ *independent of* f. *Then*

$$\int f^2 \log \frac{f^2}{\int f^2 d\mu} \leq \varepsilon \int |\mathbb{T}f|^2 d\mu + C'(\varepsilon) \int f^2 d\mu \tag{15}$$

with $C'(\varepsilon) \sim const \log \frac{1}{\varepsilon}$ *for small* ε.

Using this and [25] we obtain the following important implication.

Corollary 7.6 (Ultracontractivity Estimate). *The semigroup $P_t \equiv e^{t\mathcal{L}}$, with $\mathcal{L} \equiv \mathbb{T} \cdot \mathbb{T}$, is ultracontractive i.e. for $t > 0$ the operator $P_t : L_1(\mu) \to L_\infty(\mu)$ is bounded. Hence*

$$P_t f(w) = \int f(\tilde{w}) h_t(\tilde{w}, w) \mu(d\tilde{w})$$

with a bounded (uniformly away from $t = 0$) smooth heat kernel $h_t(\tilde{w}, w)$.

Generally it is an interesting challenge to find estimates on the heat kernel. Since our generator include jump type part, a nice method of [25] may not work. In [78] we proposed a strategy for Heisenberg group based on estimating the moments of coordinate functions and arguments of A. Grigor'yan [49]. It should be possible to generalise that to a class of free nilpotent Lie groups. First of all Gussian exponential bounds in horizontal variables could be achieved via Aronson arguments and one may hope that the bounds in other directions could be reduced to the former via a technique involving generalised integration by parts formula. Given estimates of moments one can find possibly optimal bounds for exponential function and with this obtain a bound of the form

$$\int e^{\beta d^\alpha} h_t(\cdot, w) d\mu \leq C(t, w) < \infty$$

with d a natural (homogeneous) distance from a given point on the underlying space and some positive constants α, β. Then one can follow [49] as follows: Using Chapman — Kolmogorov property

$$h_t(w', w) = \int h_{t/2}(w', \tilde{w}) h_{t/2}(\tilde{w}, w) \mu(d\tilde{w})$$

and elementary inequality

$$0 \leq -\frac{1}{C} d^\alpha(w', w) + \frac{\beta}{2}(d^\alpha(w', \tilde{w}) + d^\alpha(\tilde{w}, w))$$

for some $C \in (0, \infty)$ (independent of the points $w, w', \tilde{w}$), together with Hölder inequality, one gets

$$h_t(w', w) \leq e^{-\frac{1}{C} d^\alpha(w', w)} \cdot$$
$$\left(\int h_{t/2}^2(w', \tilde{w}) e^{\beta d^\alpha(w', \tilde{w})} \mu(d\tilde{w})\right)^{1/2} \cdot \left(\int h_{t/2}^2(\tilde{w}, w) e^{\beta d^\alpha(w, \tilde{w})} \mu(d\tilde{w})\right)^{1/2}$$
$$\leq e^{-\frac{1}{C} d^\alpha(w', w)} \hat{c}_t \cdot$$
$$\left(\int h_{t/2}(w', \tilde{w}) e^{\beta d^\alpha(w', \tilde{w})} \mu(d\tilde{w})\right)^{1/2} \cdot \left(\int h_{t/2}(\tilde{w}, w) e^{\beta d^\alpha(w, \tilde{w})} \mu(d\tilde{w})\right)^{1/2}$$

where $\hat{c}_t$ is the constant from the ultracontractivity estimate. Given such an upper bound one could possibly use arguments of [9] to obtain a lower bound. *One may conjecture that at least in the case of free nilpotent Lie groups, the bounds should have a Gaussian character in a suitable distance (possibly away from reflection hyperplanes). It would be an interesting question what is possible in more general cases and how to quantify the corresponding heat kernel bounds. Finally, is it possible to obtain sharpened version at least with a tight exponential factor on both sides of the sandwich.*

8. Further Examples with Nilpotent Lie Groups

8.1. Nonhomogenous Nilpotent Lie Groups

In this section we present an example of the structures discussed above but in case of nonhomogeneous Lie group.

Let Φ and Ψ be strictly increasing odd bijective real functions on $\mathbb{R}$ with $\Phi'(0) = 1 = \Psi'(0)$. We consider the following Lie group action on $\mathbb{H}_{\Phi,\Psi} \cong \mathbb{R}^3$

$$w \circ \tilde{w} \equiv (\Phi^{-1}(\Phi(x) + \Phi(\tilde{x})), \Psi^{-1}(\Psi(y) + \Psi(\tilde{y})), z + \tilde{z} + \Phi(x)\Psi(y)).$$

In this setup we have the following (left invariant) fields

$$Xf(w) \equiv \frac{d}{d\varepsilon} f(w \circ (\varepsilon, 0, 0))_{|\varepsilon=0} = \frac{1}{\Phi'(x)} \partial_x f(w)$$

$$Yf(w) \equiv \frac{d}{d\varepsilon} f(w \circ (0, \varepsilon, 0))_{|\varepsilon=0} = (\frac{1}{\Psi'(y)} \partial_y + \Phi(x)\partial_z) f(w)$$

and associated reflections

$$\sigma_X(w) \equiv w \circ (\Phi^{-1}(-2\Phi(x)), 0, 0) = (-x, y, z)$$

$$\sigma_Y(w) \equiv w \circ (0, \Psi^{-1}(-2\Psi(y)), 0) = (x, -y, z - 2\Phi(x)y)$$

for which we have the following fundamental relations satisfied

$$X(f \circ \sigma_X) = -(Xf) \circ \sigma_X; \qquad Y(f \circ \sigma_Y) = -(Yf) \circ \sigma_Y$$

$$= Z \equiv \partial_z$$

and the following operator play a role of dilation operator

$$D \equiv \sinh(x)X + \sinh(y)Y + 2zZ.$$

In the present case we propose to consider the following DeMazur operators

$$A_X(f) = \kappa \, \frac{f - f \circ \sigma_X}{\Phi(x)}; \qquad A_Y(f) = \kappa \, \frac{f - f \circ \sigma_X}{\Psi(y)}.$$

Now for

$$T_X \equiv X + A_X; \qquad T_Y \equiv Y + A_Y.$$

we have

$$T_X(f \circ \sigma_X) = -(T_X f) \circ \sigma_X; \qquad T_Y(f \circ \sigma_Y) = -(T_Y f) \circ \sigma_Y.$$

and with a measure

$$d\nu \equiv \Phi'(x)(\Phi(x))^{2\kappa}\Phi'(y)(\Phi(y))^{2\kappa} dxdydz$$

the following formulas of integration by parts hold

$$\int (T_X f) g d\nu = -\int f(T_X g) d\nu, \qquad \int (T_Y f) g d\nu = -\int f(T_Y g) d\nu.$$

Hence one can build up an interesting analysis in this case. We recall that a special case the group with $\Phi(x) = \sinh(x), \Psi(y) = y$ one can find e.g. in [15] (Ex.1.2.17). The DeMazur operators in this case could be related to models with $\sinh^{-2}$ interactions (for the theory of completely integrable systems see e.g. in [37; 65]).

8.2. Nilpotent Lie Groups with more general $\mathbb{T}$

Here we provide an example where number of A_σ's are necessary in a natural way. We consider generators of the form

$$X_j \equiv \partial_{x_j} + \sum_{k\neq j; k\leq m} \sum_{m+1\leq l\leq n} B_{jk;l}\, x_k \partial_{x_l}$$

with some constants $B_{jk;l} \neq 0$, for $j = 1, \dots, m$ and all $k \neq j$, $l = m+1, \dots, n$. Then each X_j admits a number of classical reflections σ_{jk}, $k \neq j$, (i.e. reflections defined by matrices with entries ± 1) which change the sign of j and k coordinates. In this case a function $x_j - x_k$ is σ_{jk} antisymmetric and can be used to define the following DeMazur operator

$$A_{jk} f \equiv \kappa \frac{f - f \circ \sigma_{jk}}{x_j - x_k}.$$

Next define

$$T_j \equiv X_j + \sum_{k\neq j; k\leq m} A_{jk}.$$

If we introduce a measure

$$d\nu \equiv \prod_{j\neq k; j,k\leq m} (x_j - x_k)^{2\kappa}\, d\lambda_n,$$

with density involving VanDerMonde determinant. Then all T_j, $j = 1, \dots, m$, satisfy integration by parts formula. One can redefine these operators by adding suitable DeMazur operators A_j corresponding to nonclassical reflections and defined with nonsymmetric functions x_j, provided we modify the measure by factors $x_j^{2\kappa}$.

There is also a more general conclusion out this example: *For any classical crystallographic group one has specific weights (as described e.g. in [36]) and a corresponding Dunkl type theory. One can use this theory for coordinates in the first strata (of coordinates of homogeneus dimension 1) and extend it to a nontrivial nilpotent Lie group case (possibly multiplying weight by suitable product if one would like to add A_σ with nonclassical reflection). This point of view suggest an interesting classification of nilpotent Lie algebras for which restriction of Cox to coordinates in the first strata conicides with a given classical crystallographic group (and for any classical crystallographic group there exists infinitely many extensions).*

9. Appendix: Functional Representation of Groups

An interesting problem is how to represent groups in terms of classes of functions with operation of composition of functions and how to classify such representations.

Such framework is rich enough to accommodate infinite discrete groups. For example, one knows that for any $n \in \mathbb{N}$, there exists an uncountable set of solutions of an equation

$$\varphi^{\circ n} = \mathbb{I}$$

in terms of continuous functions mapping a unit interval into itself, (cf. [53]).

To illustrate the possibilities we provide here a functional representation of the cellebrated Baumschalg — Soliltar group which is defined as folows, (cf. [18])

$\{a, b : a^{-1}b^m a = b^n\}$.

By a direct computation one can show that such the group can be realised by

$a = \log^{\circ j} \circ p \circ \exp^{\circ j}(t)$

$b = \log^{\circ k} \circ q \circ \exp^{\circ k}(t)$

with $k = j + 1$. Then

$a^{-1} = \log^{\circ j} \circ p^{-1} \circ \exp^{\circ j}(t)$

$b^m = \log^{\circ k} \circ q^m \circ \exp^{\circ k}(t)$.

And for $k \geq j$ we have

$a^{-1}b^m a = \log^{\circ j} \circ p^{-1} \circ \log^{\circ(k-j)} \circ q^m \circ \exp^{\circ(k-j)} \circ p \circ \exp^{\circ j}(t)$.

If we choose $k = j + 1$. Then we have

$$\begin{aligned} a^{-1}b^m a &= \log^{\circ j} \circ p^{-1} \circ \log \circ q^m \circ \exp \circ p \circ \exp^{\circ j}(t) \\ &= \log^{\circ j} \circ \log \circ q^{mp^{-1}} \circ \exp \circ p^{-1} p \circ \exp^{\circ j}(t) \\ &= \log^{\circ k} \circ (q^{mp^{-1}}) \circ \exp^{\circ k}(t) = b^{mp^{-1}} \end{aligned}$$

That is the group is realised for given $m \in \mathbb{N}$ with $n \equiv mp^{-1}$ provided p devides m, although it is possible that the noninteger values of n could be realised in the framework of functional equation theory.

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КРИСТАЛЛОГРАФИЧЕСКИЕ ГРУППЫ ДЛЯ ПОЛЕЙ ХЕРМАНДЕРА

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Аннотация. Это предварительная статья о кристаллографических группах полей Хермандера. Мы описываем картину, возникающую в анализе расширенных групп. В частности, мы вводим понятие и приводим примеры кристаллографических групп, связанных с системой полей Хермандера, а также обсуждаем некоторые связанные вопросы анализа.

Ключевые слова: расширенные группы Ли, некоммутативные операторы типа Данкла, полугруппы Маркова, оценки ядер уравнений теплопроводности и энтропии.