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η-RICCI SOLITONS AND GRADIENT RICCI SOLITONS ON *f*-KENMOTSU MANIFOLDS

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Abstract. The aim of the present research article is to discuss the f-Kenmotsu manifolds with respect to a semi-symmetric non- metric connection conceding an η-Ricci soliton and gradient Ricci soliton. Moreover, we prove that the second order symmetric tensor is a constant multiple of the metric tensor and parallel with respect to the semi-symmetric non-metric connection. In addition, we illustrate an example to exhibit that 3-dimensional f-Kenmotsu manifolds with a semi-symmetric non-metric connection concede an expanding η-Ricci soliton. Finally, it is shown that locally ϕ-symmetric 3-dimensional f-Kenmotsu manifolds with a semi-symmetric non-metric connection concede a gradient Ricci soliton.

Key words: η-Ricci Solitons, gradient Ricci solitons, f-Kenmotsu manifold, semi-symmetric non metric connection, η-Einstein manifold.

Introduction

The concept of f-Kenmotsu manifold, where f is a real constant, appears for the first time in the paper of D. Jannsens and L. Vanhecke [12]. More recently, Z. Olszak and R. Rosca [15] defined and studied the f-Kenmotsu manifold by the formula (17), where f is a function on M such that $df \wedge \eta = 0$. Here, η is the dual 1-form corresponding to the characteristic vector field ξ of an almost contact metric structure on M. The condition $df \wedge \eta = 0$ follows in fact from (17) if $\dim M \geq 5$. This does not hold in general if $\dim M = 3$.

In 1924, A. Friedmann and J.A. Schouten [10] introduced the idea of a semi-symmetric linear connection. A linear connection ∇ is said to be a semi-symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = \eta(Y)X - \eta(X)Y, \tag{1}$$

where η is a 1-form. The connection ∇ is symmetric if the torsion tensor T vanishes, otherwise, it is non-symmetric. The connection ∇ is metric connection if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi — Civita connection. Some properties of semi-symmetric non-metric connection were studied M. Ahmad et al. and Siddiqi et al. in (see [1; 2; 17; 19]) respectively.

In 1988, R. Hamilton [11] and in 2003, G. Perelman [16] studied the solution of the Poincare conjecture in dimension 3 have produced a flourishing activity in the research of self similar solutions, or solitons, of the Ricci flow. The study of the geometry of solitons, in particular their classification in dimension 3, has been essential in providing a positive answer to the conjecture; however, in higher dimension and in the complete, possibly noncompact case, the understanding of the geometry and the classification of solitons seems to remain a desired goal for a not too proximate future. In the generic case a soliton structure on the Riemannian manifold (M, g) is the choice of a smooth vector field X on M and a real constant λ satisfying the structural requirement

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda g, \tag{2}$$

where Ric is the Ricci tensor of the metric g and $\mathcal{L}_X g$ is the Lie derivative of this latter in the direction of X. In what follows we shall refer to λ as to the soliton constant. The soliton is called expanding, steady or shrinking if, respectively, $\lambda > 0$, $\lambda = 0$ or $\lambda > 0$. When X is the gradient of a potential $\psi \in C^\infty(M)$, the soliton is called a gradient Ricci soliton and the previous equation (1) takes the form

$$\nabla\nabla\psi = S + \lambda g, \qquad \text{where} \qquad Hess\psi = \nabla\nabla\psi. \tag{3}$$

Both equations (1) and (2) can be considered as perturbations of the Einstein equation

$$\mathrm{Ric} = \lambda g. \tag{4}$$

and reduce to this latter in case X or $\nabla\psi$ are Killing vector fields. When $X = 0$ or ψ is constant we call the underlying Einstein manifold a trivial Ricci soliton.

R. Sharma [18] initiated the study of Ricci solitons in contact Riemannian geometry . After that, Tripathi [20], Nagaraja et. al. [14] and others like C.S. Bagewadi et. al. and also M. Turan et al. ([3], [21]) extensively studied Ricci solitons in manifolds with different structures.

In 2009, J.C. Cho and M. Kimura introduced the notion of η-Ricci soliton [9] which was treated by C. Calin and M. Crasmareanu on Hopf hypersurfaces in complex space forms [6]. A Riemannian manifold (M, g) is called a η-Ricci soliton if there exist a smooth vector field ξ such that the Ricci tensor satisfies the following equation [7]

$$2S + \mathcal{L}_\xi g + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{5}$$

where $\mathcal{L}_\xi$ is the Lie derivative operator along the vector field ξ, S is the Ricci tensor and λ, μ are real constants. If $\mu = 0$, then η-Ricci soliton becomes Ricci soliton.

Later η-Ricci solitons in para-Kenmotsu manifolds [5] and Lorentzian para-Sasakian manifolds [4] have been studied by A. M. Blaga et al. In [7] C. Calin and M. Crasmareanu studied Ricci soliton in f-Kenmotsu manifols . Moreover, in [22] A. Yildiz et al. studied

studied 3-dimenional f-Kenmotsu and Ricci-soliton. Recently in 2018, D. Chakraborty et al. [8] studied Ricci soliton on 3-diemnensional β-Kenmotsu manifold with respect to Schouten-van Kampen connection. Motivated by these studies in the present paper, we study η-Ricci solitons and gradient Ricci soliton in f-Kenmotsu manifold with a semi-symmetric non-metric connection.

1. Preliminaries

Let M be a 3-dimensional differentiable manifold with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and Riemannian metric g such that

$$\phi^2 = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \tag{6}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \tag{7}$$

for all $X, Y \in \chi(M)$. Also, for an almost contact manifold M, it follows that [13]

$$\nabla_X \phi Y = (\nabla_X \phi)Y + \phi(\nabla_X Y), \tag{8}$$

$$(\nabla_X \eta)Y = \nabla_X \eta(Y) - \eta(\nabla_X Y). \tag{9}$$

Let R be Rieammian curvature tensor, S Ricci curvature tensor, Q Ricci operator and $\{e_1, e_2, ...e_n\}$ be orthonormal basis of M. For all $X, Y \in \chi(M)$ it follow that

$$S(X, Y) = \sum_{i=1}^{n} g(R(e_i, X)Y, e_i), \tag{10}$$

$$QX = -\sum_{i=1}^{n} (R(e_i, X)e_i \tag{11}$$

and

$$S(X, Y) = g(QX, Y). \tag{12}$$

If the Ricci tensor S of an f-kenmostsu manifol M satisfies the condition

$$S(X, Y) = ag(X, X)Y + b\eta(X)\eta(Y), \tag{13}$$

where a, b are scalars, then M is said to be η-*Einstein* manifold. If $b = 0$, then M is called *Einstein* manifold. In a 3-dimensional Riemannian manifold the curvature tensor R is defined as

$$R(X, Y)Z = S(Y, Z)X - g(X, Z)QY + g(Y, Z)QX - S(X, Z)Y, \tag{14}$$

$$-\frac{r}{2}[g(Y, Z)X - g(X, Z)Y],$$

where S is the Ricci tenosr, Q is the Ricci operator and r is the scalar curvature for 3-dimensional manifold M.

On the other hand, let M be an n-dimensional Riemannian manifold with the Riemannian connection ∇. A linear connection $\bar{\nabla}$ on M is said to be semi-symmetric metric connection if its torsion tensor $\bar{T}$ of the connection $\bar{\nabla}$ satisfies

$$\bar{T}(X,Y) = \eta(Y)X - \eta(X)Y, \tag{15}$$

where η is non-zero 1-form and $\bar{T} \neq 0$.

Moreover, $\bar{\nabla}g = 0$ then the connection is called a semi-symmetric metric connection. If $\bar{\nabla}g \neq 0$ then the connection is called a semi-symmetric non-metric connection [10].

For $n \geq 1$, M is called projectively flat if and only if the well known projective curvature tensor p vanishes. Projective curvature tensor P is defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y] \tag{16}$$

for any $X, Y, Z \in \chi(M)$, where R is the curvature tensor and S is the Ricci tensor of M [13]. $P(X,Y)\xi = 0$ for any $X, Y \in \chi(M)$, then manifold M is called ξ-projective flat [12].

2. f-Kenmotsu manifolds

Let M be a 3-dimensional almost contact manifold. (M, ϕ, ξ, η, g) is an f-Kenmotsu manifold if the covariant differentiation of ϕ satisfies [12],

$$(\nabla_X \phi)Y = f(g(\phi X, Y) - \eta(Y)\phi X) \tag{17}$$

where $f \in C^\infty(M)$ such that $df \wedge \eta = 0$. If $f = \beta = \text{const} \neq 0$, the manifold is said to be an β-Kenmotsu. If $f = 1$, then 1-Kenmotsu manifold is also called Kenmotsu manifold. If $f^2 + f' \neq 0$, then f-Kenmotsu manifold is said to be regular, where $f' = \xi f$ [12]. By using (6) and (7), it can be shown that

$$(\nabla_X \eta)Y = fg(\phi X, \phi Y). \tag{18}$$

From (3.1), we have

$$\nabla_X \xi = f(X - \eta(X)\xi). \tag{19}$$

Also from (2.8), in 3-dimensional f-Kenmotsu we have

$$R(X,Y)Z = (\frac{r}{2} + 2f^2 + 2f')(X \wedge Y) - \tag{20}$$

$$-(\frac{r}{2} + 3f^2 + 3f')[\eta(X)(\xi \wedge Y)Z + \eta(Y)(X \wedge \xi)Z]$$

and

$$S(X,Y) = (\frac{r}{2} + 2f^2 + 2f')g(X,Y) - (\frac{r}{2} + 3f^2 + 3f')\eta(X)\eta(Y). \tag{21}$$

Thus from (21), we get

$$S(X,\xi) = -2(f^2 + f')\eta(X) \tag{22}$$

Using (20) and (21), we obtain

$$R(X,Y)\xi = -(f^2 + f')[\eta(Y)X - \eta(X)Y], \tag{23}$$

$$R(\xi,X)\xi = -(f^2 + f')[\eta(X)\xi - X], \tag{24}$$

$$QX = (\frac{r}{2} + 2f^2 + 2f')X - (\frac{r}{2} + 3f^2 + 3f')\eta(X)\xi. \tag{25}$$

It can be easily checked out from (16) by using (23) and (22) that a 3-dimensional f-Kenmotsu manifold is always ξ-projectively flat [12].

3. f-Kenmotsu manifolds with semi-symmetric non-metric connection

Let $\bar{\nabla}$ be a linear connection and ∇ be a Riemannian connection of an f-Kenmotsu manifold M. This $\bar{\nabla}$ linear connection defined by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X \tag{26}$$

where η-1-form and any vector fields $X, Y \in \chi(M)$, denotes the semi-symmetric non-metric connection [10].
For f-Kenmotsu manifold with the semi-symmetric non-metric connection, using (7), (17) and (26) we have

$$\bar{\nabla}_X \phi)Y = f(g(\phi X, \phi Y)\xi - 2\eta(X)\phi X) \tag{27}$$

for any vector fields $X, Y \in \chi(M)$, where ϕ is $(1,1)$ tensor filed, ξ is a vector filed, η is a 1-form and $f \in C^{\infty}$ such that $df \wedge \eta = 0$. As consequence of $df \wedge \eta = 0$, we get

$$df = f' \text{ and } X(f) = f'\eta(X) \tag{28}$$

where $f' = \xi f$. If $f = \beta = constant \neq 0$, then the manifold is a β-Kenmotsu [13]. If $f = 0$, then the manifold is cosymplectic manifold. An f-Kenmotsu manifold with the semi-symmetric non-metric connection is said to be regular if $f^2 + f + 2f' \neq 0$.
By using (6) and (27), we get

$$\bar{\nabla}_X \xi = f(2X - \eta(X)\xi). \tag{29}$$

From (7), (26) and (27), we have

$$(\bar{\nabla}_X \eta)Y = fg(\phi X, \phi Y). \tag{30}$$

The curvature tensor $\bar{R}$ of an f-Kenmotsu manifold M with respect to the semi-symmetric non-metric connection $\bar{\nabla}$ is defined by

$$\bar{R}(X,Y)\xi = \bar{\nabla}_X \bar{\nabla}_Y \xi - \bar{\nabla}_Y \bar{\nabla}_X \xi - \bar{\nabla}_{[X,Y]}\xi. \tag{31}$$

With the help of (26), (29) and (19), we get

$$\begin{gathered}\bar{\nabla}_X \bar{\nabla}_Y \xi = X(f)2Y - X(f)\eta(Y)\xi + 2f\nabla_X Y - fX\eta(Y)\xi - \eta(Y)f^2 X \\ +\eta(Y)\eta(X)f^2\xi + f\eta(Y)X\end{gathered} \tag{32}$$

and

$$\begin{gathered}-\bar{\nabla}_{[X,Y]}\xi = -2f\nabla_X Y - 2f\eta(Y)X + 2f\nabla_Y X \\ +2f\eta(X)Y + fX\eta(Y)\xi - fY\eta(X)\xi.\end{gathered} \tag{33}$$

Using (32) and (33) in (30), we get

$$\begin{gathered}\bar{R}(X,Y)\xi = X(f)2Y - X(F)\eta(Y)\xi - Y(f)2X + Y(f)\eta(X)\xi + f^2\eta(X)Y \\ -f^2\eta(Y)X + f\eta(X)Y - f\eta(Y)X.\end{gathered} \tag{34}$$

using (28) in (34), it follows that

$$\bar{R}(X,Y)\xi = -(f^2 + f + 2f')[\eta(Y)X - \eta(X)Y]. \tag{35}$$

From (35), we have

$$\bar{R}(\xi, Y)\xi = -(f^2 + f + 2f')[\eta(Y)\xi - Y], \tag{36}$$

and

$$\bar{R}(X, \xi)\xi = -(f^2 + f + 2f')[X - \eta(X)\xi]. \tag{37}$$

Taking the inner product with Z in (4.10), we have

$$g(\bar{R}(X, Y)\xi, Z) = -(f^2 + f + 2f')[\eta(Y)g(X, Z) - \eta(X)g(Y, Z)] \tag{38}$$

which is used in the proof of the following lemma.

Lemma 1. *Let M be a 3-dimensional f-Kenmotsu manifold with semi-symmetric non-metric connection, $\bar{S}$ Ricci curvature tensor and $\bar{Q}$ Ricci operator. Then*

$$\bar{S}(X, \xi) = -2(f^2 + f + 2f')\eta(X), \tag{39}$$

$$\bar{Q}\xi = -2(f^2 + f + 2f')\xi. \tag{40}$$

Proof. Contracting with Y and Z in (38) and taking summation over $i = 1, 2....n$, from (10) expression the proof (39) is completed. then also using (2.7) and (6) in (39), the proof of (40) is completed.

Lemma 2. *Let M be a 3-dimensional f-Kenmotsu manifold with semi-symmetric non-metric connection, r scalar curvature tensor, $\bar{S}(X, Y)$ Ricci curvature tensor and $\bar{Q}X$ Ricci operator. Then it follows that*

$$\bar{S}(X, Y) = (\frac{r}{2} + f^2 + f + 2f')g(X, Y) - (\frac{r}{2} + 3f^2 + 3f + 6f')\eta(X)\eta(Y) \tag{41}$$

and

$$\bar{Q}X = (\frac{r}{2} + f^2 + f')X - (\frac{r}{2} + 3f^2 + 3f')\eta(Y)\xi. \tag{42}$$

Proof. Contracting (37) with Y, we get

$$g(\bar{R}(X, \xi)\xi, Y) = -(f^2 + f + 2f')(g(X, Y) - \eta(X)\eta(Y)) \tag{43}$$

Using (39), putting $X = \xi, Y = X, Z = Y$ in (14) and contracting with ξ, we obtain

$$\bar{R}(\xi, X, Y, \xi) = \bar{S}(X, Y) - 2(f^2 + f + 2f')g(X, Y) - \frac{r}{2}(g(X, Y) - \eta(X)\eta(Y)) \tag{44}$$

$$+2(f^2 + f + 2f')\eta(X)\eta(Y) + 2f^2 + f + 2f')\eta(X)\eta(Y).$$

With the help of (43) and (44) proff of (41) is completed.

Using (41) and (12), its verified that

$$g(\bar{Q}X - [(\frac{r}{2} + f^2 + f + 2f')X - (\frac{r}{2} + 3f^2 + 3f + 6f')\eta(X)\xi, Y] = 0. \tag{45}$$

Since $Y \neq 0$ in (45), the proof of (42) is completed.

4. η-Ricci solitons on *f*-Kenmotsu manifold with semi-symmetric non-metric connection

Fix h a symmetric tensor field of $(0,2)$-type which we suppose to be parallel with respect to the semi-symmetric non-metric connection $\bar{\nabla}$ that is $\bar{\nabla}h = 0$. Applying the Ricci commutation identity [11].

$$\bar{\nabla}^2 h(X,Y;Z,W) - \bar{\nabla}^2 h(X,Y;W,Z) = 0, \tag{46}$$

we obtain the relation

$$h(\bar{R}(X,Y)Z,W) + h(Z,\bar{R}(X,Y)W) = 0. \tag{47}$$

Realacing $Z = W = \xi$ in (47) and using (35) and also use the symmetry of h, we have

$$-(f^2+f+2f')[\eta(Y)h(X,\xi) - \eta(X)h(Y,\xi)] - (f^2+f+2f')[\eta(Y)h(\xi,\xi) - h(Y,\xi)] \tag{48}$$

Putting $X = \xi$ in (48) and by virtue of (6), we obtain By using regularity condition in (49), we have

$$-(f^2+f+2f')[\eta(Y)h(\xi,\xi) - h(Y,\xi)] = 0. \tag{49}$$

Suppose $-(f^2+f+2f') \neq 0$, it results

$$h(Y,\xi) = \eta(Y)h(\xi,\xi). \tag{50}$$

Now, we can call a regular f-Kenmotsu manifold with semi-symmetric non-metric connection with $-(f^2+f+2f') \neq 0$, where regularity, means the non-vanishing of the Ricci curvature with respect to the generator of f-Kenmotsu manifold with semi-symmetric non-metric connection.

Differentiating (50) covariantly with respect to X, we have

$$(\bar{\nabla}_X h)(Y,\xi) + h(\bar{\nabla}_X Y,\xi) + h(Y,\bar{\nabla}_X \xi) = [g(\bar{\nabla}_X Y,\xi) + g(Y,\bar{\nabla}_X \xi)]h(\xi,\xi) \tag{51}$$

$$+\eta(Y)[(\bar{\nabla}_X h)(Y,\xi) + 2h(\bar{\nabla}_X \xi,\xi)].$$

By using the parallel condition $\bar{\nabla}h = 0$, $\eta(\bar{\nabla}_X \xi) = 0$ and by the virtue of (50) in (51), we get

$$h(Y,\nabla_X \xi) = g(Y,\nabla_X \xi)h(\xi,\xi).$$

Now using (29) in the above equation, we get

$$h(X,Y) = g(X,Y)h(\xi,\xi), \tag{52}$$

which together with the standard fact that the parallelism of h implies that $h(\xi,\xi)$ is a constant, via (51). Now by considering the above equations, we can gives the conclusion:

Theorem 1. *Let (M,ϕ,ξ,η,g) be an f-Kenmotsu manifold with semi-symmetric non-metric connection with non-vanishing ξ-sectional curvature and endowed with a tensor field $h \in \gamma(T_2^0(M))$ which is symmetric and ϕ-skew-symmetric. If h is parallel with respect to $\bar{\nabla}$ then it is a constant multiple of the metric tensor g.*

Definition 1. *Let (M, ϕ, ξ, η, g) be an f-Kenmotsu manifold with semi-symmetric non-metric connection. Consider the equation [9]*

$$\mathcal{L}_\xi g + 2\bar{S} + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{53}$$

where $\mathcal{L}_\xi$ is the Lie derivative operator along the vector field ξ, S is the Ricci curvature tensor field of the metric g, and λ and μ are real constants. Writing $\mathcal{L}_\xi g$ in terms of semi-symmetric non-metric connection ∇, we obtain:

$$2\bar{S}(X,Y) = -g(\bar{\nabla}_X \xi, Y) - g(X, \bar{\nabla}_Y \xi) - 2\lambda g(X,Y) - 2\mu\eta(X)\eta(Y), \tag{54}$$

for any $X, Y \in \chi(M)$.

The data (g, ξ, λ, μ) which satisfy the equation (5.8) is said to be η- *Ricci soliton on M [9] ; in particular if $\mu = 0$ (g, ξ, λ) is Ricci soliton [9] and its called shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$ respectively [9].*

Now, from (29), the equation (54) becomes:

$$\bar{S}(X,Y) = -(2f + \lambda)g(X,Y) + (f - \mu)\eta(X)\eta(Y). \tag{55}$$

The above equations yields

$$\bar{S}(X,\xi) = -(f + \lambda + \mu)\eta(X), \tag{56}$$

$$\bar{Q}X = -(2f + \lambda)X + (f - \mu)\xi, \tag{57}$$

$$\bar{Q}\xi = -(f + \lambda + \mu)\xi, \tag{58}$$

$$\bar{r} = -\lambda n - (n-1)f - \mu, \tag{59}$$

where r is the scalar curvature. Off the two natural situations regrading the vector field V: $V \in Span\,\xi$ and $V \perp \xi$, we investigate only the case $V = \xi$.

Our interest is in the expression for $\mathcal{L}_\xi g + 2\bar{S} + 2\mu\eta \otimes \eta$. A direct computation gives

$$\mathcal{L}_\xi g(X,Y) = 2f[2g(X,Y) - \eta(X)\eta(Y)]. \tag{60}$$

In 3-dimensional f-Kenmotsu manifold with semi-symmetric non-metric connection the Riemannian curvature tensor is given by

$$\bar{R}(X,Y)Z = g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y + \bar{S}(Y,Z)X - \bar{S}(X,Z)Y - \tag{61}$$

$$-\frac{r}{2}[g(Y,Z)X - g(X,Z)Y].$$

Putting $Z = \xi$ in (61) and using (35), (39) and (42) for 3-dimensional f-Kenmotsu manifold with semi-symmetric non-metric connection, we get

$$-(f^2 + f + 2f')[\eta(Y)X - \eta(X)Y] = \eta(Y)[(\frac{r}{2} + f + 2f')X - (\frac{r}{2} + 3f^2 + 3f')\eta(X)\xi] - \tag{62}$$

$$-\eta(X)[(\frac{r}{2} + f + 2f')Y - (\frac{r}{2} + 3f^2 + 3f')\eta(Y)\xi] - 2(f^2 + f + 2f')\eta(Y)X +$$

$$+2(f^2 + f + 2f')\eta(X)Y - \frac{r}{2}[\eta(Y)X - \eta(X)Y].$$

Again, putting $Y = \xi$ in the (62) and using (7) and condition of regularity we obtain

$$\bar{Q}X = \Big[\frac{r}{2} + (\frac{r}{2} + f^2 + f') - (f^2 + f + 2f')\Big] X + \tag{63}$$

$$+ \Big[\frac{r}{2} + (\frac{r}{2} + f^2 + f') - 3(f^2 + f + 2f')\Big] \eta(X)\xi.$$

From (63), we have

$$\bar{S}(X, Y) = \Big[\frac{r}{2} + (\frac{r}{2} + f^2 + f') - (f^2 + f + 2f')\Big] g(X, Y) + \tag{64}$$

$$+ \Big[\frac{r}{2} + (\frac{r}{2} + f^2 + f') - 3(f^2 + f + 2f')\Big] \eta(X)\eta(Y).$$

Equation (64) shows that a 3-dimensional f-Kenmotsu manifold with semi-symmetric non-metric connections η-$Einstein$.

Next, we consider the equation

$$h(X, Y) = (\mathcal{L}_\xi g)(X, Y) + 2S(X, Y) + 2\mu\eta(X)\eta(Y). \tag{65}$$

By Using (61) and (64) in (65), we have

$$h(X, Y) = (r - 4f - f')g(X, Y) + (r - 4f + 5f' - 2f^2)\eta(X)\eta(Y) + 2\mu\eta(X)\eta(Y). \tag{66}$$

Putting $X = Y = \xi$ in (66), we get

$$h(\xi, \xi) = 2[r + 2f' - f^2 + \mu]. \tag{67}$$

Now, (52) becomes

$$h(X, Y) = 2[r + 2f' - f^2 + \mu]g(X, Y). \tag{68}$$

From (65) and (68), it follows that g is an η-Ricci soliton.

Therefore, we can state as:

Theorem 2. *Let (M, ϕ, ξ, η, g) be a 3-dimensional f-Kenmotsu manifold with semi-symmetric non-metric connection, then (g, ξ, μ) yields an η-Ricci soliton on M.*

Let V be pointwise collinear with ξ. i.e., $V = b\xi$, where b is a function on the 3-dimensional f-Kenmotsu manifold with semi-symmetric non-metric connection. Then

$$g(\bar{\nabla}_X b\xi, Y) + g(\bar{\nabla}_Y b\xi, X) + 2\bar{S}(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0$$

or

$$bg((\bar{\nabla}_X \xi, Y) + (Xb)\eta(Y) + bg(\bar{\nabla}_Y \xi, X) + (Yb)\eta(X) +$$

$$+2\bar{S}(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0.$$

Using (29), we obtain

$$bg(f(2X - \eta(X)\xi, Y) + (Xb)\eta(Y) + bg(f(2Y - \eta(Y)\xi, X) +$$

$$+(Yb)\eta(X) + 2\bar{S}(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0.$$

which yields

$$4bfg(X, Y) - 2bf\eta(X)\eta(Y) + (Xb)\eta(Y) + \tag{69}$$

$$+(Yb)\eta(X) + 2\bar{S}(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

Replacing Y by ξ in (5.24), we obtain

$$(Xb) + (\xi b)\eta(X) + 2bf\eta(X) - 4(f^2 + f + 2f')\eta(X) + 2\lambda\eta(X) + 2\mu\eta(X)\eta(Y). \quad (70)$$

Again putting $X = \xi$ in (70), we obtain

$$\xi b = 2(f^2 + f + 2f') - bf - \lambda - \mu.$$

Plugging this in (70), we get

$$(Xb) + 2[2(f^2 + f + 2f') - bf - \lambda + \mu]\eta(X) = 0,$$

or

$$db = \{2(f^2 + f + 2f') - bf - \lambda - \mu\}\,\eta. \quad (71)$$

Applying d on (71), we get $\{2(f^2 + f + 2f') - bf - \lambda - \mu\}\,d\eta$. Since $d\eta \neq 0$ we have

$$\{2(f^2 + f + 2f') - bf - \lambda - \mu\} = 0. \quad (72)$$

Equation(72) in (71) yields b as a constant. Therefore from (69), it follows that

$$\bar{S}(X,Y) = -(\lambda + 2bf)g(X,Y) + (bf - \mu)\eta(X)\eta(Y),$$

which implies that M is of constant scalar curvature for constant f. This leads to the following:

Theorem 3. *If in a 3-dimensional f-Kenmotsu manifold with semi-symmetric non-metric connection the metric g is an η-Ricci soliton and V is positive collinear with ξ, then V is a constant multiple of ξ and g is an η-Einstein manifold and constant scalar curvature provided bf is a constant.*

Example 1. 3-dimensional f-Kenmotsu manifold with semi-symmetric-non-metric connection:

Consider the three dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3 \,|z \neq 0\}$, where (x,y,z) are the cartesian coordinates in $\mathbb{R}^3$ and let the vector fields are

$$e_1 = z^2\frac{\partial}{\partial x}, \qquad e_2 = z^2\frac{\partial}{\partial y}, \qquad e_3 = \frac{\partial}{\partial z},$$

where e_1, e_2, e_3 are linearly independent at each point of M. Let g be the Riemannain metric defined by

$g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1$, $g(e_1,e_3) = g(e_2,e_3) = g(e_1,e_2) = 0$,

Let η be the 1-form defined by $\eta(X) = g(X,e_3)$ for any vector field X on M,

and ϕ be the (1,1) tensor field defined by $\phi(e_1) = -e_2$, $\phi(e_2) = e_1$, $\phi(e_3) = 0$.

Then by using the linearity of ϕ and g, we have $\phi^2 X = -X + \eta(X)\xi$, with $\xi = e_3$. Further $g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$ for any vector fields X and Y on M. Hence for $e_3 = \xi$, the structure defines an (δ)-almost contact structure in R^3. Let ∇ be the Levi — Civita connection with respect to the metric g, then we have

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z])$$

$$- g(Y,[X,Z]) + g(Z,[X,Y]),$$

which is know as Koszul's formula.

$$\nabla_{e_i} e_i = -\frac{2}{z} e_3, \qquad \nabla_{e_i} e_3 = -\frac{2}{z} e_i, i = 1,2; \tag{73}$$
$$\nabla_{e_1} e_2 = \nabla_{e_2} e_1 = \nabla_{e_3} e_1 = \nabla_{e_3} e_2 = \nabla_{e_3} e_3 = 0.$$

Here ∇ be the Levi — Civita connection with respect to the metric g , then we have

$$[e_1, e_2] = 0, \qquad [e_1, e_3] = -\frac{2}{z} e_1, \qquad [e_2, e_3] = -\frac{2}{z} e_2.$$

Now consider at this example for semi-symmetric non-metric connection from (26) and (73),

$$\bar{\nabla}_{e_i} e_i = -\frac{2}{z} e_3, \nabla_{e_i} e_3 = -\frac{2}{z} e_i, \quad i = 1,2, \tag{74}$$
$$\nabla_{e_i} e_j = \bar{\nabla}_{e_3} e_i = 0, \qquad \bar{\nabla}_{e_3} e_3 = e_3,$$

where $i \neq j$. We know that

$$\bar{R}(X,Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z. \tag{75}$$

By using (74) and (75) we obtain the components of the Riemann and the Ricci curvature tensor fields are computed as follows:

$$\bar{R}(e_i, e_3) e_3 = (1 - \frac{6}{Z^2}) e_i, \bar{R}(e_i, e_j) e_3 = 0, \tag{76}$$

$$\bar{R}(e_i, e_j) e_j = (\frac{2}{z} - \frac{4}{z^2}) e_i, \qquad \bar{R}(e_i, e_3) e_j = 0, \bar{R}(e_3, e_i) e_i = (\frac{2}{z} - \frac{6}{z^2}) e_3,$$

where $i \neq j = 1,2$.

From the equation (76) we can also obtain

$$\bar{R}(e_1, e_3) e_3 = (1 - \frac{6}{Z^2}) e_1,\ \bar{R}(e_2, e_3) e_3 = (1 - \frac{6}{Z^2}) e_1,\ \bar{R}(e_1, e_2) e_2 = (\frac{2}{z} - \frac{4}{Z^2}) e_1, \tag{77}$$

$$\bar{R}(e_3, e_1) e_1 = (\frac{2}{z} - \frac{6}{Z^2}) e_3,\ \bar{R}(e_3, e_2) e_2 = (\frac{2}{z} - \frac{6}{Z^2}) e_3,\ \bar{R}(e_2, e_1) e_1 = (\frac{2}{z} - \frac{4}{Z^2}) e_3,$$

Therefore, we have

$$\bar{S}(e_i, e_i) = \bar{S}(e_2, e_2) = -\frac{10}{z^2} + \frac{2}{z} + 1, i = 1,2, S(e_3, e_3) = -\frac{12}{z^2} + \frac{4}{z}, \tag{78}$$

$$\bar{S}(e_1, e_1) = \bar{S}(e_2, e_2) = -\frac{10}{z^2} + \frac{2}{z} + 1, \tag{79}$$

for $i = 1,2$. Hence M is also an *Einstein* manifold. In this case, from (54) (e_i, e_i) follows, for

$$f[2g(e_i, e_i) - \eta(e_i)\eta(e_i)] + 2\bar{S}(e_i, e_i) + 2\lambda g(e_i, e_i) + 2\mu\eta(e_i)\eta(e_i),$$

$$2f(2 - \delta_{ij}) + 2(-\frac{10}{z^2} + \frac{2}{z} + 1) + 2\lambda + 2\mu\delta_{ij},$$

for all $i \in \{1,2,3\}$. Therefore, we have $\lambda = (2f - \frac{2}{z} - \frac{5}{z^2} + 1)$ and $\mu = (3f - \frac{4}{z} - \frac{5}{z^2} + 1)$, where $f = \frac{1}{2}(\frac{5}{z} + \frac{11}{z^2} + 2)$ the he data (g, ξ, λ, μ) is an η-Ricci soliton on (M, ϕ, ξ, η, g) with respect to the semi-symmetric non-metric connection is expanding.

5. Gradient Ricci soliton in f-Kenmotsu manifolds with semi-symmetric non metric connection

If the vector field V is the gradient of a potential function-ψ then the gradient Ricci soliton with semi-symmetric non metric connection and equation (19) assume the form

$$\bar{\nabla}\bar{\nabla}\psi = \bar{S} + \lambda g. \tag{80}$$

This reduces to

$$\bar{\nabla}_Y D\psi = \bar{Q} + \lambda Y \tag{81}$$

where D denotes the gradient operator of g. Form (81) it follows

$$\bar{R}(X,Y)D\psi = (\bar{\nabla}_X Q)Y - (\bar{\nabla}_Y Q)X. \tag{82}$$

Differentiating (42) we have

$$(\bar{\nabla}_W Q)X = \frac{dr(W)}{2}(X - \eta(X)\xi)) - (\frac{r}{2} - 3f^2 + 3f'))+ \tag{83}$$

$$+(fg(W,X) - \eta(X)\eta(W)) + \eta(X)\bar{\nabla}_W\xi.$$

In (83) replacing $W = \xi$, we obtain

$$(\bar{\nabla}_\xi Q)X = \frac{dr(\xi)}{2}(X - \eta(X)\xi)). \tag{84}$$

Then we have

$$g((\bar{\nabla}_\xi Q)X - (\bar{\nabla}_X Q)(\xi,\xi) = \tag{85}$$

$$= g(\frac{dr(\xi)}{2}(X - \eta(X)\xi,\xi)) = \frac{dr(\xi)}{2}(g(X,\xi) - \eta(X)) = 0.$$

Using (85) and (84), we obtain

$$g(\bar{R}(\xi,X)D\psi,\xi) = 0. \tag{86}$$

From (36)

$$g(\bar{R}(\xi,Y)D\psi,\xi) = -(f^2 + f + 2f')(g(Y,D\psi) - \eta(Y)\eta(D\psi)).$$

Using (86), we get

$$-(f^2 + f + 2f')(g(Y,D\psi) - \eta(Y)\eta(D\psi)) = 0$$

$$-(f^2 + f + 2f')(g(Y,D\psi) - \eta(Y)g(D\psi,\xi)) = 0,$$

or

$$(g(Y,D\psi) - g(Y,\xi)g(D\psi,\xi)) = 0,$$

which implies

$$D\psi = (\xi\psi)\xi, \quad \text{since} \quad -(f^2 + f + 2f') \neq 0. \tag{87}$$

Using (87) and (81)

$$\bar{S}(X,Y) + \lambda g(X,Y) = g(\bar{\nabla}_Y D\psi, X) = g(\bar{\nabla}_Y(\xi\psi)\xi, X) =$$

$$= (\xi\psi)g(\bar{\nabla}_Y\xi, X) + Y(\xi\psi)\eta(X) =$$

$$= (\xi\psi)g(f(2Y - \eta(Y)\xi, X) + Y(\xi\psi)\eta(X).$$

$$\bar{S}(X,Y) + \lambda g(X,Y) = -2f(\xi\psi)g(Y,X) - f(\xi\psi)\eta(Y)\eta(X) + Y(\xi\psi)\eta(X). \tag{88}$$

Putting $X = \xi$ in (88) and using (39) we get

$$\bar{S}(Y,\xi) + \lambda\eta(Y) = Y(\xi\psi) = [\lambda - 2(f^2 + f + 2f') - f(\xi\psi)]\eta(Y). \tag{89}$$

Interchanging X and Y in (88), we get

$$\bar{S}(X,Y) + \lambda g(X,Y) = 2f(\xi\psi)g(Y,\phi X) - f(\xi\psi)\eta(X)\eta(Y) + X(\xi\psi)\eta(Y). \tag{90}$$

Adding (88) and (90) we get

$$2\bar{S}(X,Y)+2\lambda g(X,Y) = 4f(\xi\psi)g(X,Y)-2f\eta(X)\eta(Y)+Y(\xi\psi)\eta(X)+X(\xi\psi)\eta(Y). \tag{91}$$

Using (89) in (91) we have

$$\begin{aligned}\bar{S}(X,Y) + \lambda g(X,Y) &= 2f(\xi\psi)[g(X,Y) - \eta(X)\eta(Y)]+ \\ &+[\lambda - 2(f^2 + f + 2f') - f(\xi\psi)]\eta(Y)\eta(X).\end{aligned} \tag{92}$$

Then using (81) we have

$$\bar{\nabla}_Y D\psi = 2f(\xi\psi)(Y - \eta(Y)\xi) + [\lambda - 2(f^2 + f + 2f') - f(\xi\psi)]\eta(Y)\xi. \tag{93}$$

Using (93) we calculate

$$\begin{aligned}\bar{R}(X,Y)D\psi &= \bar{\nabla}_X\bar{\nabla}_Y D\psi - \bar{\nabla}_Y\bar{\nabla}_X D\psi - \bar{\nabla}_{[X,Y]}D\psi = \\ &= 2fX(\xi\psi)Y - 2fY(\xi\psi)X+ \\ &+2fY(\xi\psi)\eta(X)\xi - 2fX(\xi\psi)\eta(Y)\xi+ \\ &+ [\lambda - 2(f^2 + f + 2f') - 2f(\xi\psi)](\nabla_X\eta)(Y)\xi - (\nabla_Y\eta)(X)\xi+ \\ &+ [\lambda - 2(f^2 + f + 2f') - 2f(\xi\psi)](\nabla_X\xi)\eta(Y)\xi - (\nabla_Y\xi)\eta(X).\end{aligned} \tag{94}$$

Taking inner product with ξ in (94), we get

$$0 = g(\bar{R}(X,Y)D\psi,\xi) = 2f[\lambda - 2(f^2 + f + 2f') - 2f(\xi\psi)]g(\phi Y, X). \tag{95}$$

Thus we have $2f[\lambda - 2(f^2 + f + 2f') - 2f(\xi\psi)] = 0$.

Now we consider the following cases:

(i) $f = 0$, or

(ii) $[\lambda - 2(f^2 + f + 2f') - 2f(\xi\psi)] = 0$,

(iii) $f = 0$ and $[\lambda - 2(f^2 + f + 2f') - 2f(\xi\psi)] = 0$.

Case (i) If $f = 0$, the manifold reduces to a f-Kenmotsu manifold with respect to a semi-symmetric non-metric connection.

Case (ii) Let $[\lambda - 2(f^2 + f + 2f') - 2f(\xi\psi)]] = 0$. If we use this in (89) we get $Y(\xi\psi) = -f(\xi\psi)\eta(Y)$. Substitute this value in (91) we obtain

$$\bar{S}(X,Y) + \lambda g(X,Y) = 2f(\xi\psi)g(X,Y). \tag{96}$$

Now, contracting (96), we get

$$\bar{r} + 3\lambda = 6f(\xi\psi), \tag{97}$$

which implies

$$(\xi\psi) = \frac{\bar{r}}{6f} + \frac{\lambda}{2f}. \tag{98}$$

If $\bar{r} = \mathrm{const}$, then $(\xi\psi) = \mathrm{const} = k(say)$. Therefore from (87) we have $D\psi = (\xi\psi)\xi = k\xi$. This we can write this equation as

$$g(D\psi, X) = k\eta(X), \tag{99}$$

which means that $d\psi(X) = k\eta(X)$. Applying d this, we get $kd\eta = 0$. Since $d\eta \neq 0$, we have $k = 0$. Hence we get $D\psi = 0$. This means that $\psi = constant$ Therefore equation (56) reduces to

$$\bar{S}(X,Y) = -2(f^2 + f + 2f')g(X,Y),$$

that is M is an *Einstein* manifold.

Case (iii) Using $f = 0$ and $[\lambda - 2(f^2 + f + 2f') - 2f(\xi\psi)] = 0$ in (89) we obtain $Y(\xi\psi) = 2f(\xi\psi)\eta(Y)$. Now as in *Case (ii)* we conclude that the manifold is an *Einstein* manifold.

Thus we have the following theorem.

Theorem 4. *If a 3-dimensional f-Kenmotsu manifold with a semi symmetric non-metric connection with constant scalar curvature admits gradient Ricci soliton, then the manifold is either Kenmotsu manifold or an Einstein manifold provided* $f, f' = \mathrm{const}$.

In [21] it was proved that if a 3-dimensional compact connected trans-Sasakian manifold is of constant curvature, then it is either α-Sasakian or β-Kenmotsu. Since for a 3-dimensional Riemannian manifold constant curvature and Einstein manifold are equivalent, therefore from the Theorem 3 we state the following statements.

Corollary 1. *If a compact 3-dimensional f-Kenmotsu manifold with a semi symmetric non-metric connection with constant scalar curvature admits Ricci soliton, then the manifold f-Kenmotsu.*

Also in [21], authors proved that a 3-dimensional connected trans-Sasakian manifold is locally ϕ-symmetric if and only if the scalar curvature is constant provided α and β are constants. Hence from Theorem 3 we obtain the following:

Corollary 2. *If a locally ϕ-symmetric 3-dimensional f-Kenmotsu manifold with a semi symmetric non-metric connection its admits gradient Ricci soliton, then manifold is either f-Kenmotsu or Einstein manifold provided* $f,\ f' = \mathrm{const}$.

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СОЛИТОНЫ η-РИЧЧИ И ГРАДИЕНТНЫЕ СОЛИТОНЫ РИЧЧИ НА *f*-КЕНМОЦУ МНОГООБРАЗИЯХ

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Аннотация. Целью настоящей статьи является изучение многообразий f-кенмоцу относительно полусимметрической неметрической связности, допускающей солитон η-Риччи и градиентный солитон Риччи. Кроме того, мы доказываем, что симметричный тензор второго порядка является постоянным кратным метрическому тензору и параллельным относительно полусимметрической неметрической связности. В дополнение мы проиллюстрировали пример, демонстрирующий, что 3-мерные f-Кенмоцу многообразия с полусимметричной неметрической связностью допускают расширяющийся η-Риччи солитон. Наконец, показано, что локально ϕ-симметричные 3-мерные f-Кенмоцу многообразия с полусимметрической неметрической связностью допускают градиентный солитон Риччи.

Ключевые слова: солитоны η-Риччи, градиентные солитоны Риччи, f-Кенмотцу многообразие, полусимметричная неметрическая связность, η-эйнштейново многообразие.